University of Alberta

Smoothness and Decompositions in Banach Spaces

by
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A Thesis
Submitted to the Faculty of Graduate Studies and Research
in Partial Fulfillment of the Requirements for the Degree
of Doctor of Philosophy

Department of Mathematics

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I dedicate this thesis, with love, to Alison

ABSTRACT

The object of this thesis is to examine the nature of differentiable structures on Banach spaces. We are particularly interested in determining the availability of functions with bounded nonempty support.

To this end, we first show that if a separable space admits a function whose k^{th} Fréchet derivative is locally Lipschitz smooth then it also admits a function which is also $k+1$ times Gateaux differentiable. In contrast we show that for Γ uncountable and $p \geq 3$ odd, $\ell_p(\Gamma)$ admits no p times Gateaux differentiable function with bounded nonempty support.

In a separate direction, we find conditions which guarantee the existence of smooth partitions of unity. First we show that a Banach space which is C^k–smooth and has a WCG dual admits C^k–smooth partitions of unity. Second, we show that a space admitting a WLUR norm which is the uniform limit on bounded sets of C^{k+1}–smooth norms admits C^k–smooth partitions of unity. These two results are quite general and apply to Banach spaces which were not previously known to admit smooth partitions of unity.

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It was not an automatic decision to continue my studies beyond the Master's level. The deciding factor was the opinion of one person, Vaclav Zizler. It is my belief that without his support I would not have finished this or any other thesis; his kindness and compassion enabled me to continue when it would have been easier to abandon my work. His is a debt that cannot be repaid.

Of the others that have helped make this thesis a reality, I would like to single out three. First, I thank Jon Vanderwerff for his comraderie and his sense of humour and for holding down the fort while I was away. In addition, his comments and proofreading of the manuscript were invaluable. Second, to René Poliquin and Henry Van Roessel I extend my profound thanks for the time and hardware that enabled me to finish on time.

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CHAPTER 1

Introduction

Our studies in this thesis will be concerned with differentiability and will be comprised of two separate thrusts. The first will examine the nature of higher order differentiability in separable Banach spaces. It is a well–known result (cf. [D], [DGZ$_2$]) that all separable Banach spaces can be equiped with a Gateaux norm, and in certain circumstances, this result can be extended to higher order differentiability. The first theorem in this direction appeared in [FWZ] where it is shown that in any separable Banach space with a norm whose Fréchet derivative is locally Lipschitz there exists a norm which has these properties of the first norm while also being twice Gateaux differentiable. A classical result of Kurzweil [K] states that for p odd the usual norm on $\ell_p(\Gamma)$ is $p-1$ times Fréchet differentiable but not p times. Recently S. Troyanski [Tr$_2$] showed that for p odd, $\ell_p(\Gamma)$ admits a p times Gateaux differentiable norm if and only if Γ is countable.

We improve these results in several ways. First, we show that similar statements to those in [FWZ] are obtainable for higher orders and we extend them to include generic functions (i.e. not necessarily a norm; of course in this case we do not obtain a norm). In addition, we show that these functions with higher order derivatives are, in a very strong sense, dense; in fact, the theorem we prove will yield approximations uniformly on bounded sets. Finally, our theorem is a strengthening of the results in [Tr$_2$] concerning the $\ell_p(\mathbb{N})$ spaces for $p \geq 3$ odd. It is interesting to note that the proof of our theorem shows that the approximations of convex functions can be realized with convex functions. In light of the theory from [BN] a corollary of our theorem is:

Proposition. *Any convex function which is bounded on bounded sets and defined*

from a separable Hilbert space to $\mathbb{R}$ *can be approximated uniformly on bounded sets by convex* C^1 *functions each of which has a Lipschitz derivative and is twice Gateaux differentiable.*

We also extend the results in [Tr_2] by showing that for Γ uncountable, $p > 1$ and p odd, $\ell_p(\Gamma)$ does not admit a p times Gateaux differentiable function which has nonempty bounded support.

The second direction of this thesis concerns partitions of unity. It is a well-known fact that all Banach spaces (indeed all metric spaces) admit partitions of unity subordinated to any open covering so it is a natural question to ask if these partitions can be chosen so that each element of each partition is of a given degree of differentiability. In addition to the intrinsic interest in these objects, we have the following important application (cf. [T]) of the existence of smooth partitions of unity. (For definitions see Chapter 2.)

Proposition. *If a Banach space* X *admits* C^k*-smooth partitions of unity, then for any normed linear space* Y *with norm* $\|\cdot\|$ *and any continuous maps* $f : X \to Y$ *and* $p : X \to (0,1)$ *there exists a* C^k*-smooth partition of unity* $\{\psi_i\}_{i\in I}$ *and a family of points* $\{w_i\}_{i\in I} \subset Y$ *such that for all* $x \in X$, $\|f(x) - \sum_{i\in I}[\psi_i(x)]w_i\| < p(x)$.

Surprisingly, the smooth partition question is not easy and it was only in 1966 that a comprehensive understanding of the situation for separable spaces was reached [BF]. We note that in the finite dimensional case one is always guaranteed C^∞-smooth partitions of unity, but from [BF] we see that given any $n \in \mathbb{N}$ it is possible to find a Banach space X which has partitions of unity which are C^n-smooth but has none which are C^{n+1}-smooth.

The situation in the non-separable case is more difficult still. It was only in 1972 that C^1-smooth partitions of unity were obtained for non-separable Hilbert

spaces [W]. The following year brought the paper of Toruńczyk [T] which completely opened the partitions question, for he was able to characterize spaces which admit C^k–smooth partitions of unity via homeomorphic embeddings into $c_0(\Gamma)$ which satisfy a coordinate–wise differentiability criterion. Using this characterization he showed that all Hilbert spaces and all $c_0(\Gamma)$ spaces admit C^∞–smooth partitions and all reflexive space have C^1–smooth partitions. Also the situation regarding the L_p spaces $p \in (1, \infty)$ was completely solved.

As we will observe later, all these spaces are weakly compactly generated and thus one is led to ask whether the results stated above extend to all WCG spaces. This question was fully solved in [GTWZ] using the theorem in [T]. Recently, new results have been obtained in [V] and [DGZ_1]. The contribution of this thesis to this theory will be to broaden the class of spaces admitting differentiable partitions of unity in two ways. In the first, we assume that the dual of a Banach space is WCG while the original space has a C^k–smooth function with bounded nonempty support and proceed to find C^k–smooth partitions. Of course it is of interest to know that this can be achieved for a non–WCG space: we note that the space constructed in [JL] shows that this is the case. The second direction dispenses with weak compactness altogether and instead requires the existence of a weak locally uniformly rotund norm which can be approximated uniformly on bounded sets by norms which are C^{k+1}–smooth away from the origin. This is enough to assert the existence of C^k–smooth partitions of unity.

We now turn to the organization of the thesis. The following chapter broadly outlines the main ideas and tools used in the subsequent proofs. There we introduce the bulk of the notations and definitions, as well as give the proofs of some elementary lemmata. In the third chapter the results concerning differentiability in separable Banach space are proved and some corollaries are given. Chapter 4

begins the study of partitions of unity while the fifth and final chapter ends it for us. The fourth chapter will be devoted to the case where there is a WCG dual while the fifth treats the other direction discussed above. Both chapters conclude with applications and open problems.

CHAPTER 2

Definitions and Setting

In this chapter we shall be interested in setting the framework for the rest of the thesis. Here we shall be content to give the important definitions, to recall theorems from the literature that will be of use to us, and to verify some simple results that we use frequently. Because the results presented here are central to the thesis, all notations given are fixed for the remainder of the work. We begin by establishing the following conventions. The symbol X will denote a Banach space, and we designate the dual of X, the set of all continuous linear functionals on X, as X^*. When we speak of the bidual of X we refer to the dual of the dual of X and this space is denoted X^{**}. The ball of radius δ centered at $x \in X$ is denoted $B(x, \delta)$ and we write $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ for the natural, rational and the real number systems respectively. Wherever possible we reserve the following letters: $w, x, y, z \in X$; $i, j, k, l, m, n, p, q \in \mathbb{N}$; and $\phi \in X^{**}$. The space X is isomorphic to a subspace of its bidual and we shall frequently use this result; in doing so, no reference is made to this isomorphism and elements of X are routinely considered as elements of X^{**}.

For a set $Y \subset X$ we shall write $\mathrm{sp}Y$ to denote the linear span of Y, i.e., the set of all finite linear combinations of elements in Y; when we wish to denote the norm, weak, or weak* closed linear span of Y we write $\overline{\mathrm{sp}}Y$, $\overline{\mathrm{sp}}^w Y$, or $\overline{\mathrm{sp}}^{w^*} Y$ respectively. The expression $T : X \to Y$ will be used to indicate that a mapping T is defined for all elements in X and takes these to elements in a Banach space Y. If $T : X \to Y$ then it is possible to define a map $T^* : Y^* \to X^*$ by $T^*(f)(x) = f(T(x))$ for each $x \in X$. Whenever a subspace $M \subset X$ is given, we can define a subspace $M^\perp = \{f \in X^* : f(x) = 0,\ x \in M\} \subset X^*$. Also, $\mathrm{dens}X$ will denote the density character of X, that is, the infimum of the cardinalities of all dense subsets of X.

For an ordinal γ we write $|\gamma|$ to denote the cardinality of γ and we let ω be the first infinite ordinal.

2.1 Projectional Resolutions of the Identity

In our investigations into the structures of Banach spaces we shall want to be able to access information about the availability of well–behaved subspaces of a given space. In Hilbert spaces all closed subspaces are again Hilbert spaces and the structure of closed subspaces of the so–called $\ell_p(\mathbb{N})$ spaces is quite well understood (cf. e.g. [LT]). For non–separable spaces the available tools for studying such problems are limited but in some rather general circumstances one very effective tool is present.

Convention: If γ is an infinite ordinal we shall write $\Gamma = 0 \cup [\omega, \gamma]$, the space of all infinite ordinals less than or equal to γ together with 0 (where $0 < \omega$, and we agree to write $0 + 1 = \omega$). We also write $\Gamma_\gamma = 0 \cup [\omega, \gamma)$.

Definition. Suppose $\text{dens}X = |\gamma|$ where γ is the smallest ordinal of cardinality $|\gamma|$. Then we define a ***projectional resolution of the identity*** (PRI) of X to be a long sequence of projections $\{T_\alpha\}_{\alpha \in \Gamma}$ on X with $T_0 = 0$ and T_γ the identity operator, such that:

i) $\|T_\alpha\| = 1$ for all $\alpha \in [\omega, \gamma]$;

ii) $T_\alpha T_\beta = T_\beta T_\alpha = T_\alpha$ for $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$;

iii) $x \in \overline{\text{sp}}\{(T_{\alpha+1} - T_\alpha)(x) : \alpha \in \Gamma_\gamma\}$ for each $x \in X$;

iv) $\text{dens}T_\alpha(X) \leq |\alpha|$ for each $\alpha \in \Gamma$; and

v) $\{\alpha \in \Gamma_\gamma : \|(T_{\alpha+1} - T_\alpha)(x)\| \geq \epsilon\}$ is finite for each $x \in X$ and $\epsilon > 0$.

Remark: Condition ii), though a usual part of the definition of a PRI will not be directly used in any part of this thesis. The inclusion here is purely formal, but this condition is implicitly involved in some of the arguments we shall encounter.

The existence of such a collection allows us to define a family of subspaces which are nested, controlled with respect to density character, and each element in the original space can be identified with a unique countable subcollection of these subspaces. As we shall note in Sections 2.3 and 2.4, the class of spaces admitting a PRI is (remarkably) quite large; even so, this basic definition will be insufficient for our purposes. Fortunately, in our circumstances, we shall be able to refine iv) in this definition.

Consider the following situation. Suppose we have a condition I on X which guarantees a projectional resolution of the identity $\{T_\alpha\}$ and suppose further that each subspace $(T_{\alpha+1} - T_\alpha)(X)$ also satisfies I. Then the space $(T_{\alpha+1} - T_\alpha)(X)$ also admits a projectional resolution of the identity and so there exists a splitting of this space into subspaces. Note, however, that the density character of each of these new subspaces (as an ordinal) is smaller than that from which it was decomposed. When we know that all subspaces of X obtained from the decomposition inherit the condition I, this process can be repeated as long as we wish. In particular, we shall want to split the space into separable pieces so a transfinite induction process is called into play. When we do this we gain the additional power of a separable decomposition, but at a cost. Conditions ii), iii), and v) are retained (with the obvious modification on the set Γ) but we can no longer ensure that the operators are uniformly bounded. We can hope, however, that each individual operator is bounded, and this leads us to the following definition.

Definition. Suppose $\text{dens}X = |\gamma|$ where γ is the smallest ordinal of cardinality $|\gamma|$. A *separable projectional resolution of the identity* (separable PRI) of X is a long sequence of projections $\{T_\alpha\}_{\alpha\in\Gamma}$ on X with $T_0 = 0$ and T_γ the identity operator, such that:

i) $\|T_\alpha\| < \infty$ for all $\alpha \in \Gamma$;

ii) $T_\alpha T_\beta = T_\beta T_\alpha = T_\alpha$ for $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$;

iii) $x \in \overline{\text{sp}}\{(T_{\alpha+1} - T_\alpha)(x) : \alpha \in \Gamma_\gamma\}$

iv) for each $\alpha \in \Gamma_\gamma$, $(T_{\alpha+1} - T_\alpha)(X)$ is separable; and

v) $\{\alpha \in \Gamma_\gamma : \|(T_{\alpha+1} - T_\alpha)(x)\| \geq (\|T_{\alpha+1}\| + \|T_\alpha\|)\epsilon\}$ is finite for each $x \in X$ and for each $\epsilon > 0$.

In general, the spaces to which we give attention will not even possess a PRI, much less a separable one. Rather, in all situations the dual spaces admit separable PRI's. Our need to ascertain information about the original space will be satisfied via investigations in the bidual, thus necessitating an examination of the dual projections, i.e., the adjoints of the elements in the PRI. To fix the situation we let X be a Banach space such that its dual X^* admits a separable PRI $\{T_\alpha\}_{\alpha \in \Gamma}$. The results which will be of interest to us are:

iii)* $\phi \in \overline{\text{sp}}^{w^*}_{\alpha \in \Gamma_\gamma}\{(T^*_{\alpha+1} - T^*_\alpha)(\phi)\}$ for each $\phi \in X^{**}$; and

iv)* for each $\alpha \in \Gamma_\gamma$, $(T^*_{\alpha+1} - T^*_\alpha)(X^{**})$ is w^*–separable.

These two results can immediately be obtained from the definitions of the adjoint and the weak* topology.

We close this section with some notation that will be used in the sequel. For a (separable) PRI $\{T_\alpha\}_{\alpha \in \Gamma}$ on X we write $\tau_\alpha = T_{\alpha+1} - T_\alpha$ and we denote for each $x \in X$, $\Lambda(x) = \{\alpha : \tau_\alpha(x) \neq 0\} \subset \Gamma_\gamma$ and for each $f \in X^*$, $\Lambda^*(f) = \{\alpha : \tau^*_\alpha(f) \neq 0\} \subset \Gamma_\gamma$. Of course $\tau^*_\alpha = T^*_{\alpha+1} - T^*_\alpha$.

2.2 Convexity and Smoothness in Banach Spaces

In this section, we introduce the notions of smoothness and convexity and present a class of spaces for which there is a separable PRI on the dual. As the theory of smoothness and convexity is extensive, we shall be content to define only

those concepts which appear in the sequel, and we shall not elaborate on the many facets of this branch of Banach spaces. One concept which will certainly be of central import to our investigations is that of the Fréchet derivative of a function $h : X \to \mathbb{R}$.

Definition. A map $h : X \to \mathbb{R}$ is *Fréchet differentiable at* x if there exists an element $h'(x) \in X^*$ such that

$$\lim_{\|y\| \to 0} \frac{h(x+y) - h(x) - h'(x)(y)}{\|y\|} = 0.$$

In this case we say that $h'(x)$ is the *Fréchet derivative of* h *at* x; when the limit exists for all $x \in X$ and is continuous in x we say that h is continuously Fréchet differentiable. Higher order derivatives are defined inductively as follows. If $h : X \to \mathbb{R}$ is $k-1$ times differentiable at x we say that h is k *times Fréchet differentiable at* x if there exists a bounded symmetric k–multilinear form $h^{(k)}(x) : X^k \to \mathbb{R}$ such that

$$\lim_{\|y_k\| \to 0} \frac{[h^{(k-1)}(x+y_k) - h^{(k-1)}(x)](y_1, \ldots, y_{k-1}) - h^{(k)}(x)(y_1, \ldots, y_k)}{\|y_k\|} = 0$$

and we say that $h^{(k)}(x)$ is the k^{th} *Fréchet derivative* of h. The notation $C^k(X, \mathbb{R})$ will be used to indicate the space of all k times continuously differentiable functions $h : X \to \mathbb{R}$.

We now can state a special form of the implicit function theorem. In the notation below h_1 and h_2 denote the first and second partial derivatives respectively of the map $h : X \times \mathbb{R} \to \mathbb{R}$.

The Implicit Function Theorem. *Let X be a Banach space and suppose $h : X \times \mathbb{R} \to \mathbb{R}$ is continuous on a neighbourhood of (x_0, α_0) with $h(x_0, \alpha_0) = 0$. If $h_2(x_0, \alpha_0)$ exists, is continuous in x and is non–zero then there is a unique*

continuous function $g : X \to \mathbb{R}$ *such that* $g(x_0) = \alpha_0$ *and a neighbourhood* U *of* x_0 *such that for all* $x \in U$, $h(x, g(x)) = 0$. *If, in addition,* $h_1(x_0, \alpha_0)$ *exists and is continuous on a neighbourhood of* (x_0, α_0) *then* $g(x) \in C^1(X, \mathbb{R})$ *and* $g'(x) = -h_1(x_0, y_0)[h_2(x_0, y_0)]^{-1}$.

Remark. It is immediate from an induction argument that if the function h is k times differentiable then the function g is differentiable of the same degree.

Definition. A function $h : X \to \mathbb{R}$ is said to be ***Gateaux differentiable at*** x if there exists $h'(x) \in X^*$ such that

$$\lim_{t \to 0} \frac{h(x + ty) - h(x) - h'(x)(ty)}{t} = 0$$

for all $y \in X$. Higher order Gateaux differentiability is defined analogously to the Fréchet case.

Definition. The ***support*** of a function $f : X \to Y$ is the norm closure of the set $\{x \in X : f(x) \neq 0\}$ and is denoted supp$f$.

Definition. A space X is called C^k*–smooth* if there exists a function $h \in C^k(X, \mathbb{R})$, not identically 0, such that supp$h$ is bounded.

Note that a norm cannot be differentiable at 0, so that when we say a norm is C^k–smooth it is understood that it is k–times Fréchet at all points different from 0. Let us observe that if X admits a norm which is C^k–smooth, then the space itself is C^k–smooth. To see this, we compose the norm with a C^∞–smooth function, defined from $\mathbb{R}$ to $\mathbb{R}$, which is constant on a neighbourhood of the origin (of $\mathbb{R}$) and is zero for all reals t such that $|t| \geq 1$. The effect of this composition is to flatten out the cusp that occurs at $x = 0$. We remark in passing that the converse to this result, i.e., that a C^k–smooth space has a C^k–smooth norm, was a long standing open question settled in the negative by Haydon [H] in the case $k = 1$.

The existence of smooth norms is an extensive field of study and the duality of smoothness and convexity is well understood (see e.g. [B],[D]). Though our interest in convexity will center on the notion of weak local uniform rotundity, the notions of strict convexity and local uniform rotundity will also enter our discussions. We now introduce these concepts.

Definition. A norm $\|\cdot\|$ is *(weakly) locally uniformly rotund* [(W)LUR] if $2\|x\|^2 + 2\|x_m\|^2 - \|x + x_m\|^2 \to 0$ implies x_m converges (weakly) to x.

Definition. A norm $\|\cdot\|$ is called *strictly convex* if $1/2\|x+y\| = \|x\| = \|y\|$ implies $x = y$.

Remark: Observe that a WLUR norm is strictly convex. Indeed, if $1/2\|x + y\| = \|x\| = \|y\|$ then by WLUR y converges weakly to x, i.e. they are equal.

As promised earlier, we shall introduce a class of spaces the duals of which admit projectional resolutions of the identity. The theory associated with these spaces is very rich, and has been the object of intense study since the 1970's. We shall be content to recall the following concepts.

Definition. A Banach space X is called an *Asplund space* if each convex, continuous, real–valued function defined on X is Fréchet differentiable on a dense G_δ subset of X.

For our purposes, we shall need the following three results concerning Asplund spaces;

1)X is Asplund if and only if every separable subspace of X has a separable dual (cf. [B]).

2) If a space X admits a C^1–smooth norm, then X is an Asplund space. This result be can proven easily via the characterization of Asplund spaces stated above

in 1). Indeed, we need only note that the map $x \to \|\cdot\|'(x)$ is norm–to–norm continuous and apply the Bishop–Phelps theorem (cf. e.g. [D]). (The norm attaining functionals on the unit sphere are precisely the derivatives of the norm.)

3) If X is an Asplund space then there exists a separable PRI on X^*. This follows from the result in [FG] together with the observation, as noted in [DGZ_1], that the splitting by the projectional resolution of the identity yields subspaces which are dual to Asplund spaces. Note that not all subspaces of the dual will share this property.

2.3 The Space $c_0(\Gamma)$ and Smooth Partitions of Unity

In the study of Banach spaces, there are various spaces of special interest. One of these is

$$c_0(\Gamma) = \{f : \Gamma \to \mathbb{R} \text{ with cardinality of } \{\gamma \in \Gamma : |f(\gamma)| \geq \epsilon\} \text{ finite for each } \epsilon > 0\}$$

endowed with the sup norm. This space and the study of well–behaved injections into it will be of central importance to this thesis. This importance is well illustrated by the role this space plays in the study of differentiable partitions of unity. To see this interplay, we recall some definitions.

Definition. A *locally finite partition of unity* on a space X is a family of functions $\{\psi_i\}_{i\in I}$, $\psi_i : X \to \mathbb{R}$ for each $i \in I$ such that;

1. $0 \leq \psi_i(x) \leq 1$ for all $i \in I$ and all $x \in X$;
2. given $x \in X$ there exists a neighbourhood V_x of x such that $\text{supp}\psi_i \cap V_x \neq \emptyset$ for only a finite number of $i \in I$;
3. $\sum_{i\in I} \psi_i(x) = 1$ for each $x \in X$.

Definition. A space X *admits C^k–smooth partitions of unity* if given any open covering $\{V_\beta\}_{\beta\in B}$ there exists a locally finite partition of unity $\{\psi_i\}_{i\in I}$ such that

for each $i \in I$, $\psi_i \in C^k(X, \mathbb{R})$ and $\text{supp}\psi_i \subset V_\beta$ for some $\beta \in B$.

In their seminal paper [BF], Bonic and Frampton addressed the question of conditions on a space X which guarantee that X admits C^k–smooth partitions of unity. In this work they outlined the basic concepts and proved that all separable spaces which are C^k–smooth admit C^k–smooth partitions of unity. The turning point in this theory came with the proof, by Toruńczyk [T], of the inter–relationship between spaces which admit smooth partitions of unity and the space $c_0(\Gamma)$. To appreciate the true power of this theorem, one need only examine the complicated proof in Wells [W] which established that non–separable Hilbert spaces admit C^1–smooth partitions of unity: C^∞–smooth partitions of unity are an immediate consequence of Toruńczyk's theorem. Before we present this theorem, some basics concerning the space $c_0(\Gamma)$ will be explored. First, if $x = (x_\alpha)_{\alpha \in \Gamma} \in c_0(\Gamma)$ then we shall refer to the value $x_\alpha \in \mathbb{R}$ as the α coordinate of x. Thus we denote by e_α the element of $c_0(\Gamma)$ which has the value 1 on the α coordinate and is 0 on the remaining ones. Note that the element e_α can be identified with an element of the dual and that the dual space $[c_0(\Gamma)]^* = \overline{\text{sp}}\{e_\alpha : \alpha \in \Gamma\}$ where this closure is taken with respect to the dual norm. Finally for each $\gamma \in \Gamma$ we define $p_\gamma : c_0(\Gamma) \to \mathbb{R}$ by $p_\gamma(f) = f(\gamma)$.

Definition. A map $h : X \to Y$ is a *homeomorphic embedding* if h is 1–1 and continuous and $h^{-1} : h(X) \to X$ is continuous.

Theorem 2.1. *(Toruńczyk). A space X admits C^k–smooth partitions of unity if and only if there exists a set Γ and a homeomorphic embedding $h : X \to c_0(\Gamma)$ such that $h_\gamma = p_\gamma \circ h \in C^k(X, \mathbb{R})$ for each $\gamma \in \Gamma$.*

This theorem has been the catalyst for much of the subsequent work on the partitions question and is central to the proofs in [GTWZ],[DGZ$_1$]. Similar to these results, the theory presented in the fourth and fifth chapters uses this theorem and

a technique of Kadets. We would like to discuss this technique here. Essentially, this aim of this technique is to show that a given function is continuous by proving that given sequences are totally bounded. Let us make this more precise.

Lemma 2.1. *Let $h : X \to Y$ be a continuous 1–1 function. To show $h^{-1} : h(X) \to X$ is continuous it suffices to show that if $h(x_k) \to h(x)$ and if $\epsilon > 0$ is given, then $\{x_k\}_{k=1}^{\infty}$ is bounded and there exists a finite dimensional subspace $F \subset X$ such that for some $N \in \mathbb{N}$, $\inf\{\|x_k - y\| : y \in F\} < \epsilon$ for all $k \geq N$.*

Proof: We want to show that $x_k \to x$; because h is 1–1 and continuous this will follow if $\{x_k\}_{k=1}^{\infty}$ is totally bounded. We fix $\epsilon > 0$ and look for a finite ϵ–net. First use the hypothesis to find a finite dimensional subspace $F \subset X$ such that for $k \geq N$

$$\inf\{\|x_k - y\| : y \in F\} < \epsilon/2.$$

Next find C such that $\|x_k\| \leq C$ for all $k \in \mathbb{N}$. Since F is finite dimensional we can find a finite set of points $\{y_1, \ldots, y_m\} \subset F$ so that $\cup_{i=1}^{m} B(y_i, \epsilon/2)$ cover the C ball of F. Then $x_1, \ldots, x_{N-1}, y_1, \ldots, y_m$ is a finite ϵ–net of the sequence $\{x_k\}_{k=1}^{\infty}$.

To conclude this section we prove the following elementary proposition which will be useful in the application of Theorem 2.1.

Lemma 2.2. *If a map $h : X \to c_0(\Gamma)$ is such that the coordinate functions h_α are equicontinuous, then h is itself continuous.*

Proof. We shall prove that if $x_n \to x$ in the norm topology of X then $h(x_n) \to h(x)$. Fix $\epsilon > 0$. Since the functions $h_\alpha : X \to \mathbb{R}$ are equicontinuous, there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies $|h_\alpha(x) - h_\alpha(y)| < \epsilon/2$. The range of h is $c_0(\Gamma)$ so there exists a finite subset $A = \{\alpha_1, \ldots, \alpha_k\} \subset \Gamma$ such that $h_\alpha(x) < \epsilon/2$ for all $\alpha \notin A$. Find $N_0 \in \mathbb{N}$ such that $n \geq N_0$ implies $\|x_n - x\| < \delta$. Each of

the functions h_{α_i}, $1 \leq i \leq k$, is continuous so there exist N_i, $1 \leq i \leq k$, such that $|h_{\alpha_i}(x_n) - h_{\alpha_i}(x)| < \epsilon$ whenever $n \geq N_i$. Putting $N = \max\{N_0, \ldots, N_k\}$ it follows immediately that for $n \geq N$ $\max\{|h_\alpha(x_n) - h_\alpha(x)| : \alpha \in \Gamma\} < \epsilon$ and so $h(x_n) \to h(x)$.

2.4 Weakly Compactly Generated Spaces

In Chapter 4 the class of weakly compactly generated (WCG) spaces is encountered and we shall develop some preliminary concepts and results concerning these spaces in this section. We begin with the definition.

Definition. A Banach space X is *weakly compactly generated* (WCG) if there is weak compact set $K \subset X$ such that $\overline{\text{sp}}K = X$.

Examples:

1. Any separable space is WCG. Indeed, take a set $\{x_k\}_{k=1}^{\infty}$ dense on the sphere of X. Then $K = \{\frac{1}{k}x_k\}_{k=1}^{\infty} \cup \{0\}$ is the desired weak compact set.
2. If X is reflexive, then the unit ball of X is the obvious candidate for K.
3. For arbitrary Γ, $c_0(\Gamma)$ is WCG. To see this, note that the elements $\{e_\gamma : \gamma \in \Gamma\}$ together with 0 constitute a weak compact set whose closed linear span generates X.

Though a great deal of research has been devoted to the study of WCG spaces, (cf. e.g. [D]) the majority of results we use in this thesis are contained in the paper of Amir and Lindenstrauss [AL] and the main results will be stated below. The one exception is this: although their work shows the existence of a projectional resolution of the identity, the need for a separable PRI was not evident until the work of Troyanski [Tr_1]. We need a variant of the result in [Tr_1] so we shall outline a proof here while making the observations necessary to obtain the variant in question. We note before starting that we impose conditions on the weak compact set which

generates the space. However, by convexifying and symmetrizing an arbitrary weak compact generating set, the conditions stated below, although appearing to be more restrictive, require only the existence of a weak compact generating set. The conditions as stated are needed to obtain the additional statement in 1) below. We first state a definition, recall the famous theorem of Amir and Lindenstrauss [AL], and then collect some of the theory associated with this theorem.

Definition. A compact set K is called an *Eberlein compact* if it is homeomorphic to a weak compact subset of some Banach space.

Theorem 2.2. *(Amir–Lindenstrauss) Let K be a convex, symmetric and weak compact subset of a non–separable space X such that $X = \overline{\text{sp}}K$. Then*
1) X admits a PRI $\{T_\alpha\}_{\alpha\in\Gamma}$ such that $(T_{\alpha+1} - T_\alpha)(K) \subset K$,
2) the dual unit ball is Eberlein compact in the w^* topology,
3) there exists a linear, bounded, one–to–one map from X^* into $c_0(\Gamma)$ for some Γ.

We immediately use this theorem to show that any WCG space admits a separable PRI with an additional property.

Lemma 2.3. *If X is non–separable with density character γ and K is a generating, symmetric, convex, weak compact subset of X, then X admits a separable PRI $\{T_\alpha\}_{\alpha\in\Gamma}$ with the added condition that there exists a family of constants $\{N_\alpha\}_{\alpha\in\Gamma_\gamma}$ such that $(T_{\alpha+1} - T_\alpha)(K) \subset N_\alpha K$ for each $\alpha \in \Gamma_\gamma$.*

Proof. To obtain the desired resolution we proceed by transfinite induction on the density character of X. If the space X is separable then there is nothing to prove. Now suppose that the density character of X is $|\gamma| > \aleph_0$ (the first infinite cardinal). Applying Theorem 2.2 yields a resolution $\{Q_\lambda\}_{\lambda\in 0\cup[\omega,\gamma]}$ with

i) $\|Q_\lambda\| = 1$ for all $\lambda \in [\omega, \gamma]$;

ii) $Q_\lambda Q_\mu = Q_\mu Q_\lambda = Q_\lambda$ for $\lambda \leq \mu$;

iii) $x \in \overline{\text{sp}}_{\lambda \in 0 \cup [\omega,\gamma)}\{(Q_{\lambda+1} - Q_\lambda)(x)\}$ for each $x \in X$;

iv) $\text{dens} Q_\lambda(X) \leq |\lambda|$ for each $\lambda \in [\omega, \gamma]$;

v) $\{\lambda \in 0 \cup [\omega,\gamma) : \|(Q_{\lambda+1} - Q_\lambda)(x)\| \geq \epsilon\}$ is finite for each $x \in X$ and $\epsilon > 0$; and

vi) $(Q_{\lambda+1} - Q_\lambda)(K) \subset K$ for each $\lambda \in 0 \cup [\omega,\gamma)$.

Define $q_\lambda = Q_{\lambda+1} - Q_\lambda$ and note that for each $\lambda < \gamma$ the subspace $q_\lambda(X)$ has density character $M_\lambda \leq \lambda + 1 < \gamma$; furthermore it is also WCG because $q_\lambda(K)$ is a weak compact subset of $q_\lambda(X)$ which generates $q_\lambda(X)$. Thus by the induction hypothesis, for each $\lambda < \gamma$ there is a PRI $\{R_\theta(\lambda)\}_{\theta \in 0 \cup [\omega, M_\lambda)}$ for $q_\lambda(X)$ and a family of constants $\{N_\theta(\lambda)\}_{\theta \in 0 \cup [\omega, M_\lambda)}$ such that $(R_{\theta+1}(\lambda) - R_\theta(\lambda))(q_\lambda(K)) \subset N_\theta(\lambda)(q_\lambda(K))$.

We now let $\Gamma = \{(\lambda, \theta) : \lambda \in 0 \cup [\omega,\gamma), \theta \in 0 \cup [\omega, M_\lambda)\}$ and we order Γ as follows: $(\lambda_1, \theta_1) < (\lambda_2, \theta_2)$ if $\lambda_1 < \lambda_2$ or if $\lambda_1 = \lambda_2$ and $\theta_1 < \theta_2$. (Note that this is a well ordering and thus Γ can be viewed as an ordinal.) Now, for each $\alpha \in \Gamma$ we define a map $T_\alpha : X \to X$ for $\alpha = (\lambda, \theta)$ by

$$T_\alpha = R_\theta(\lambda) \circ (Q_{\lambda+1} - Q_\lambda) + Q_\lambda.$$

The proof that this construction yields a separable PRI can be found in [D]. The final condition of the lemma follows from the fact that

$$(T_{\alpha+1} - T_\alpha)(K) = (R_{\theta+1}(\lambda) - R_\theta)(Q_{\lambda+1} - Q_\lambda)(K) \subset N_\theta(\lambda)K$$

and so we can set $N_\alpha = N_\theta(\lambda)$. This finishes the proof.

To end this chapter, we introduce two more definitions and discuss them briefly. We begin with:

Definition. A set S is called *angelic* if for any set $D \subset S$, the closure of D in S is equal to the sequential closure of D in S, i.e. for each element d in the closure of D there exists a sequence $\{d_n\}_{n=1}^\infty \subset D$ converging to d (in the topology on S).

The relevance of this concept to our program has been hinted at in preceding discussions. The only result we need concerning angelicity and WCG spaces is that of Eberlein [E] which states that Eberlein compact spaces are angelic. (Of course Eberlein never stated this in such terms!)

Definition. A collection $(x_\alpha, f_\alpha)_{\alpha\in\Gamma} \subset X \times X^*$ is called a *Markuševič basis* (M–basis) if $f_\alpha(x_\beta) = 1$ when $\alpha = \beta$ and $f_\alpha(x_\beta) = 0$ for $\alpha \neq \beta$, $X = \overline{\text{sp}}\{x_\alpha\}_{\alpha\in\Gamma}$ and $\{f_\alpha\}_{\alpha\in\Gamma}$ is total on X; a M–basis is called *shrinking* if $X^* = \overline{\text{sp}}\{f_\alpha\}_{\alpha\in\Gamma}$.

Although not necessary for the development of the thesis, the following connections between M–bases and WCG spaces are listed; firstly, a Banach space X is WCG if and only if X has a M–basis $(x_\alpha, f_\alpha)_{\alpha\in\Gamma}$ such that $\{x_\alpha\}_{\alpha\in\Gamma} \cup \{0\}$ is weak compact and secondly, X is a C^1–smooth WCG space if and only if X has a shrinking M–basis.

CHAPTER 3

Higher Order Differentiability in Banach Spaces

3.1. Introduction

This chapter will begin with the examination of higher order Gateaux differentiability in separable spaces. The starting point of these investigations will be the work [FWZ] where the following theorem is proved:

Theorem 3.1. *If X is a separable Banach space which admits a differentiable norm whose derivative is (locally) Lipschitz on the unit sphere, then X admits a norm which is twice Gateaux differentiable and whose first derivative is also (locally) Lipschitz on the unit sphere.*

Using the techniques developed in the proof of this theorem, the following section extends this result to higher order differentiability and to more general functions than norms. In addition, we also obtain an approximation result, as described in Theorem 3.2.

In the third section, we show that the phenomenon of higher Gateaux differentiability in separable spaces does not extend to the non–separable case. Indeed, we strengthen the result in [Tr_2] by showing that there is no p times Gateaux differentiable function with bounded nonempty support on $\ell_p(\Gamma)$ for $p > 1$ odd and Γ uncountable.

3.2. The Separable Case

To prepare the way for the development of this theory, we begin with the statement and proof of the following lemma.

Lemma 3.1. *If the k^{th} derivative of $f : X \to \mathbb{R}$ is locally Lipschitz and Z is a*

norm relatively compact subset of X then for each $x \in X$ there is a constant $\delta_x > 0$ such that $f^{(k)}$ is Lipschitzian on $B(x, \delta_x) - Z$.

Proof. The proof is elementary in nature and follows from standard compactness arguments.

Theorem 3.2. *Let X be a separable Banach space which admits a function f which is Lipschitz on bounded sets, is k times differentiable, and whose k^{th} derivative is (locally) Lipschitz. Then for $r > 0$ and $\epsilon > 0$ given, X admits a function $g \in C^k(X, \mathbb{R})$, whose k^{th} derivative is (locally) Lipschitz and Gateaux differentiable, such that $|g(x) - f(x)| < \epsilon$ whenever $\|x\| \leq r$. Furthermore, if the function f is a k times differentiable norm then the function g can be chosen to be an equivalent norm whose k^{th} derivative is (locally) Lipschitz on the unit sphere.*

Proof. We shall prove the locally Lipschitz case, the Lipschitz case being similar. To begin the proof we fix $\epsilon > 0$, find $C \in \mathbb{R}$ such that f is Lipschitz with constant C on $B(0, r+1)$, and select a set $\{x_i\}_{i=1}^{\infty}$ dense on the unit sphere of X. Next, fix a function $\psi_1 : \mathbb{R} \to \mathbb{R}$ such that ψ_1 is non–negative and even, vanishes outside $[\frac{-\epsilon}{2C}, \frac{\epsilon}{2C}]$ and satisfies $\int_{\mathbb{R}} \psi_1 = 1$. Setting $f_0 = f$ and $\psi_{n+1} = 2^n \psi_1(2^n t)$ for $t \in \mathbb{R}$, $n \geq 1$, we define a sequence of functions $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ by

$$f_n(x) = \int_{\mathbb{R}^n} f_0(x - \sum_{i=1}^{n} t_i x_i) \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_n.$$

We shall show that this sequence of functions converges uniformly on bounded sets to a function g which satisfies

i) $\sup_{\|x\| \leq r} \{|f(x) - g(x)|\} \leq \epsilon$,

ii) g is k times Fréchet differentiable and its k^{th} derivative is locally Lipschitz,

iii) g is $k+1$ times Gateaux differentiable.

First we observe that each f_n is continuous. Now if $\|x\| \leq s$, we fix a number

c_{s+1} so that $|f(z) - f(y)| \leq c_{s+1}\|z - y\|$ whenever $\|z\| \leq s+1$ and $\|y\| \leq s+1$. Then, for $m > n$,

$$\begin{aligned}|f_m(x) - f_n(x)| &= \left| \int_{R^m} \left[f_0(x - \sum_{i=1}^{m} t_i x_i) - f_0(x - \sum_{i=1}^{n} t_i x_i) \right] \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_m \right| \\ &\leq \int_{|t_i| \leq \epsilon/(2^i C)} \| \sum_{i=n+1}^{m} t_i x_i \| c_{s+1} \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_m \\ &\leq c_{s+1} \epsilon / (2^n C)\end{aligned}$$

so by the Cauchy criterion, $f_n \to g$ uniformly on $\{x : \|x\| \leq s\}$ for some $g : X \to \mathbb{R}$. Setting $n = 0$, $s = r$ and letting m tend to infinity we have $c_{r+1} = C$ and thus $|g(x) - f(x)| < \epsilon$ for $\|x\| \leq r$. This proves i).

To prove ii), we shall show that $g^{(k)}(x)(h_1, \ldots, h_k) = \lim_n f_n^{(k)}(x)(h_1, \ldots, h_k)$ for each $x \in X$ uniformly on a small ball centered at x. With this aim in mind we first show that each f_n is k times Fréchet differentiable,

$$f_n^{(j)}(x)(h_1, \ldots, h_j) = \int_{\mathbb{R}^n} f_0^{(j)}(x - \sum_{i=1}^{n} t_i x_i)(h_1, \ldots, h_j) \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_n \quad (3-1)$$

for each $j \leq k$, and each of these derivatives is locally Lipschitz. We fix x and proceed by induction. The case $j = 0$ is immediate from the definition of these functions so we assume that the formula is true for the case $j - 1 < k$. Then

$$\begin{aligned}&\frac{f_n^{(j-1)}(x + h_j)(h_1, \ldots, h_{j_1}) - f_n^{(j-1)}(x)(h_1, \ldots, h_{j_1})]}{\|h_j\|} \\ &= \int_{\mathbb{R}^n} \frac{[f_0^{(j-1)}(x + h_j - \sum_{i=1}^{n} t_i x_i) - f_0^{(j-1)}(x - \sum_{i=1}^{n} t_i x_i)](h_1, \ldots, h_{j-1})}{\|h_j\|} \\ &\qquad \cdot \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_n\end{aligned}$$

so by taking limits as $\|h_j\| \to 0$ we see that (3-1) is true for each $n \in \mathbb{N}$ and for each $j \leq k$. The set $\cup_{n=1}^{\infty} \{\sum_{i=1}^{n} t_i x_i : |t_i| \leq \epsilon/(2^i C)\}$ is compact, so by using Lemma 3.1

and the fact that $f_0^{(j)}$ is locally Lipschitz for each $j \leq k$ we obtain a number $\delta_x^j > 0$ such that $f_0^{(j)}$ is locally Lipschitz on

$$B(x, \delta_x^j) - \cup_{n=1}^{\infty} \{\sum_{i=1}^{n} t_i x_i : |t_i| \leq \epsilon/(2^i C)\} \qquad (3-2)$$

with Lipschitz constant C_x^j. Thus if $y, z \in B(x, \delta_x^j)$ then

$$|[f_n^{(j)}(y) - f_n^{(j)}(z)](h_1, \ldots, h_j)| \leq \int_{\mathbb{R}^n} C_x^j \|y - z\| \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_n = C_x^j \|y - z\|$$

and so the k^{th} derivatives of the functions f_n are uniformly locally Lipschitz.

Now if $y \in B(x, \delta_x^j)$ then using (3–2) we have for $m > n$ and $h_l \in X$ with $\|h_l\| \leq 1$, $1 \leq l \leq j$, that

$$\begin{aligned}
&|f_m^{(j)}(y) - f_n^{(j)}(y)(h_1, \ldots, h_j)| \\
&\qquad = \int_{\mathbb{R}^m} [f_0^{(j)}(y - \sum_{i=1}^{m} t_i x_i) - f_0^{(j)}(y - \sum_{i=1}^{n} t_i x_i)](h_1, \ldots, h_j) \\
&\qquad\quad \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_m \\
&\qquad = \int_{\mathbb{R}^m} [f_0^{(j)}([y - \sum_{i=n+1}^{m} t_i x_i] - \sum_{i=1}^{n} t_i x_i)(h_1, \ldots, h_j) \\
&\qquad\quad - f_0^{(j)}(y - \sum_{i=1}^{n} t_i x_i)(h_1, \ldots, h_j)] \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_m \\
&\qquad \leq \int_{\mathbb{R}^m} C_x^j \| \sum_{i=n+1}^{m} t_i x_i \| \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_m \leq C_x^j \epsilon/(2^n C).
\end{aligned}$$

We obtain $\|f_m^{(j)}(y) - f_n^{(j)}(y)\| \leq C_x^j \epsilon/(2^n C)$ by taking the supremum over all $(h_1, \ldots, h_j) \in (B_X)^n$. This estimate is valid for all $y \in B(x, \delta_x^j)$ as $n \to \infty$ so $g^{(j)}$ exists in the Fréchet sense and is locally Lipschitz. We see from the above estimation that

$$g^{(j)}(y)(h_1, \ldots, h_j) = \lim_n \int_{\mathbb{R}^n} f_0^{(j)}(y - \sum_{i=1}^{n} t_i x_i)(h_1, \ldots, h_j) \prod_{i=1}^{n} \psi_i(t_i) dt_1 \cdots dt_n.$$

This proves ii).

The proof of iii) is as follows. Fix $l \in \mathbb{N}$ with $1 \leq l \leq m$ and fix $x, h_1, \ldots, h_k \in X$. We first show that $f_m^{(k)}(x)(h_1, \ldots, h_k)$ is Gateaux differentiable in the direction x_l. Indeed, by using a change of variables below we have

$$\begin{aligned}
&\lim_{\alpha \to 0} \int_{\mathbb{R}^m} \frac{1}{\alpha} [f_0^{(k)}(x + \alpha x_l - \sum_{i=1}^m t_i x_i)(h_1, \ldots, h_k) - f_0^{(k)}(x - \sum_{i=1}^n t_i x_i)(h_1, \ldots, h_k)] \\
&\qquad \cdot \prod_{i=1}^m \psi_i(t_i) dt_1 \cdots dt_m \\
&= \lim_{\alpha \to 0} \int_{\mathbb{R}^m} \frac{1}{\alpha} \Big[f_0^{(k)}(x - \sum_{\substack{i=1 \\ i \neq l}}^m t_i x_i - (t_l - \alpha) x_l)(h_1, \cdots, h_k) \\
&\qquad - f_0^{(k)}(x - \sum_{i=1}^m t_i x_i)(h_1, \ldots, h_k) \Big] \cdot \prod_{i=1}^m \psi_i(t_i) dt_1 \cdots dt_m \\
&= \lim_{\alpha \to 0} \int_{\mathbb{R}^m} f_0^{(k)}(x - \sum_{i=1}^m t_i x_i)(h_1, \ldots, h_k) \psi_1(t_1) \cdots \frac{\psi_l(t_l + \alpha) - \psi_l(t_l)}{\alpha} \\
&\qquad \cdots \psi_m(t_m) dt_1 \cdots dt_m \\
&= \int_{\mathbb{R}^m} f_0^{(k)}(x - \sum_{i=1}^m t_i x_i)(h_1, \ldots, h_k) \psi_1(t_1) \cdots \psi_l'(t_l) \cdots \psi_m(t_m) dt_1 \cdots dt_m
\end{aligned}$$

Because $\int_{\mathbb{R}} \psi_i'(t) dt = c < \infty$ it follows (similar to above) that for $i \leq n \leq m$,

$$|f_m^{(k+1)}(x)(h_1, \ldots, h_k, x_l) - f_n^{(k+1)}(x)(h_1, \ldots, h_k, x_l)| \leq c C_x^k \epsilon / (2^n C)$$

and so for $x, h_1, \ldots, h_k \in X$

$$\begin{aligned}
g^{(k+1)}(x)(h_1, \ldots, h_k, x_l) = \lim_m \int_{\mathbb{R}^m} & f_0^{(k)}(x - \sum_{i=1}^m t_i x_i)(h_1, \ldots, h_k) \\
& \psi_1(t_1) \cdots \psi_l'(t_l) \cdots \psi_m(t_m) dt_1 \cdots dt_m
\end{aligned}$$

for each $l \in \mathbb{N}$.

Since the $g^{(k)}$ is locally Lipschitz and $\{x_i\}_{i=1}^\infty$ is a dense set of directions,

$$g^{(k+1)}(y)(h_1, \ldots, h_k, h) = \lim_m g^{(k+1)}(y)(h_1, \ldots, h_k, x_{i_m})$$

uniformly for $\|h_l\| \leq 1$, $1 \leq l \leq k$ and $y \in B(x, \delta_x)$ whenever $x_{i_m} \to h$. In addition $g^{(k+1)}(y)(h_1, \ldots, h_k, x_i)$ is locally Lipschitz in y for each i and thus for each choice of h_i and k, $g^{(k+1)}(y)(h_1, \ldots, h_k, k)$ is continuous in y. It follows that $g^{(k+1)}$ is a $k+1$ multilinear form for each fixed x.

In the case where the function f is a norm we set $f_0(x) = \|x\|^2$. Then, proceeding as in the proof above, we see further that the functions f_n are convex and even. Indeed, $\psi_n \geq 0$ for each $n \in \mathbb{N}$, and so

$$f_n(1/2(x+y)) = \int_{\mathbb{R}^n} \|(1/2(x+y) - \sum_{i=1}^{n} t_i x_i)\|^2 \prod_{i=1}^{\infty} \psi_i(t_i) dt_1 \cdots dt_n$$

$$\leq \int_{\mathbb{R}^n} [1/2\|(x - \sum_{i=1}^{n} t_i x_i)\|^2 + 1/2\|(x - \sum_{i=1}^{n} t_i x_i)\|^2] \prod_{i=1}^{\infty} \psi_i(t_i) dt_1 \cdots dt_n.$$

It follows that the function g as obtained above is also convex and even. The Lipschitz constant for the function f_0 on the ball of radius s is $2s + \frac{\epsilon}{C}$. Without loss of generality we shall assume $r \geq 5$ and that $\epsilon < C$. If we set $B = \{x \in X : g(x) \leq 16\}$ then we claim that the Minkowski functional of B, $\mu_B(x) = \inf_\alpha \{x/\alpha \in B\}$, is an equivalent norm which is k times Fréchet differentiable and its k^{th} derivative is locally Lipschitz and is Gateaux differentiable.

To prove this claim we first observe that $g(0) \leq \epsilon/C$ and if $\|x\| > 5$ then $\mu(x) > 1$, therefore the Minkowski functional on the set B defines an equivalent norm. Note that $\mu : X \to \mathbb{R}$ is defined implicitly as the function satisfying the equation $h(x, \mu(x)) = 0$, where $h(x, \beta) : X \times \mathbb{R} \to \mathbb{R}$ is given by $h(x, \beta) = g(\frac{x}{\beta}) - 16$. The first partial derivative of h, $h_1(x, \beta) = \beta^{-1} g'(\frac{x}{\beta})$, exists in the Fréchet sense. We can find the first k Fréchet derivatives of μ at the point (x, β) by applying the implicit function theorem, provided $h_2(x, \beta) \neq 0$. We shall prove that this is the case for all $x \in X$, $\|x\| \geq 5$. Note that a simple calculation via the chain rule yields

$$h_2(x, \beta) = -g'(\frac{x}{\beta})(\frac{x}{\beta^2}) = -\frac{1}{\beta} g'(\frac{x}{\beta})(\frac{x}{\beta})$$

so we would like to show that $g'(x)(x) \neq 0$ outside some ball. With this aim, we use the fact that for $t_i \leq \epsilon/(2^i C)$, $\| \sum_{i=1}^n t_i x_i \| \leq \epsilon/C$ and consequently for $\|x\| \geq 5$

$$\begin{aligned}
&\frac{1}{\alpha}(\|(1+\alpha)x - \sum_{i=1}^n t_i x_i\|^2 - \|x - \sum_{i=1}^n t_i x_i\|^2) \\
&= \frac{1}{\alpha}(\|(1+\alpha)x - \sum_{i=1}^n t_i x_i\| + \|x - \sum_{i=1}^n t_i x_i\|)(\|(1+\alpha)x - \sum_{i=1}^n t_i x_i\| - \|x - \sum_{i=1}^n t_i x_i\|) \\
&\geq \frac{4}{\alpha}(\|(1+\alpha)x - \sum_{i=1}^n t_i x_i\| - \|x - \sum_{i=1}^n t_i x_i\|) \\
&\leq \frac{4}{\alpha}(\|(1+\alpha)(x - \sum_{i=1}^n t_i x_i) + \alpha \sum_{i=1}^n t_i x_i\| - \|x - \sum_{i=1}^n t_i x_i\|) \\
&\leq \frac{4}{\alpha}(\|(1+\alpha)(x - \sum_{i=1}^n t_i x_i)\| - \alpha\|\sum_{i=1}^n t_i x_i\| - \|x - \sum_{i=1}^n t_i x_i\|) \\
&= 4(\|(x - \sum_{i=1}^n t_i x_i)\| - \|\sum_{i=1}^n t_i x_i\|) \geq 12.
\end{aligned}$$

This holds for all $N \in \mathbb{N}$ so we have for such x

$$g'(x)(x) = \lim_n \int_{\mathbb{R}^n} f_0(x - \sum_{i=1}^n t_i x_i) \prod_{i=1}^n \psi_i(t_i) dt_1 \cdots dt_n \geq 12.$$

Thus the implicit theorem applies and

$$\mu'(x) = -[h_2(x, \mu(x))]^{-1}[h_1(x, \mu(x))] = [g'\Big(\frac{x}{\mu(x)}\Big)\Big(\frac{x}{\mu(x)}\Big)]^{-1} g'\Big(\frac{x}{\mu(x)}\Big).$$

holds for $\|x\|^2 > 16$. The fact that this formula holds outside some ball implies that it is true for all $x \neq 0$ because $\mu(x)$ is a norm. In addition, $|g(x) - \|x\|^2| \leq \epsilon$ for all $\{x : \|x\| \leq 5\}$ so in particular if $\mu(x) = 1$ then $\|x\| \leq 5$ and $|16 - \|x\|^2| < \epsilon$. By homogeneity $|4\mu(x) - \|x\|| < \epsilon$ so taking as our norm $|||x||| = 4\mu(x)$ we obtain the desired estimate. When restricted to finite dimensional subspaces, $g^{(k+1)}$ exists in the Fréchet sense and is continuous so it follows that $|||\cdot|||$ is a $k+1$ times Gateaux differentiable norm such that $|||\cdot|||^k$ is locally Lipschitz. This concludes the proof of the theorem.

As noted in Chapter 1, we obtain the following corollaries:

Corollary 3.1. *(Troyanski) For each $p \geq 3$ odd, the Banach space $\ell_p(\mathbb{N})$ admits a norm which is p times Gateaux differentiable.*

Corollary 3.2. *Any convex function which is bounded on bounded sets and defined from a separable Hilbert space to $\mathbb{R}$ can be approximated uniformly on bounded sets by convex C^1 functions each of which has a Lipschitz derivative and is twice Gateaux differentiable.*

Proof: This follows immediately from Theorem 3.2 and the result in [BN] which states that in a Hilbert space every convex function which is bounded on bounded sets can be approximated uniformly on bounded sets by convex functions which are Fréchet differentiable and whose derivatives are Lipschitz.

3.3. The Non–separable Case

Before stating and proving the main result of this section, we recall the following two results from [FPWZ].

Fact 1. If $\delta(\epsilon)$ is the modulus of smoothness of the norm $\|\cdot\|$ of X and if $u, h \in X$ and $\|\cdot\|'(-u)(h) = 0$ then $\|th - u\| \geq \|u\|(1 + \delta(\frac{\|h\|}{2}))$.

Fact 2. Let Γ be uncountable and suppose P is a polynomial of degree less than p on $\ell_p(\Gamma)$. Then there exists $y \in \ell_p(\Gamma)$ such that $P(ty) = 0$ for all real t.

Theorem 3.3. *For Γ uncountable and $p \geq 3$ odd, the space $\ell_p(\Gamma)$ admits no p times Gateaux differentiable function which has bounded nonempty support.*

Proof: Suppose such a function f exists. Let $X = \ell_p(\Gamma)$ and without loss of generality we can suppose that $\text{supp} f \subset \frac{1}{4}B_X$ and $f(0) = \sqrt{2}$. If we set $\psi(x) = \frac{1}{f(x)^2}$ then ψ is an extended real valued lower semicontinuous function defined on X. Now consider the non–empty set

$$L_0 = \{(x, \psi(x)) \in X \times \mathbb{R} : \psi(x) \leq 2\}.$$

Since ψ is lower semicontinuous L_0 is closed in $X \times \mathbb{R} = Y$.

We norm the space Y by $|||(x,\alpha)||| = (\|x\|^p + |\alpha|^p)^{\frac{1}{p}}$ for each $(x,\alpha) \in Y$. The space Y with this norm is isometrically isomorphic to $\ell_p(\Gamma)$. We use the result of [As] to see that the set of points in Y having farthest points in L_0 is dense in Y. Find such a point (x_0,α_0) with $\|x_0\| < \frac{1}{4}$ and $\alpha_0 > 5$ having farthest point $(z_0,\psi(z_0))$. Note that $\psi(0) = \frac{1}{2}$, so $\psi(z_0) < 1$. To see this, observe that if (l,λ) is farthest then $\|x_0 - l\|^p + |\alpha_0 - \lambda|^p \geq \|x_0 - 0\|^p + (\alpha_0 - \frac{1}{2})^p$. Because $p > 2$ and $\alpha_0 > 5$, this forces $\lambda < 1$.

We now translate by this farthest point. We obtain a point $u = (x,\alpha) = (x_0 - z_0, \alpha_0 - \psi(z_0))$, $|||u||| = r$, which has $(0,0)$ as a farthest point in $L = L_0 - (z_0,\psi(z_0))$ and we write $\phi(z) = \psi(z_0 + z) - \psi(z_0)$. We see easily that ϕ is p times Gateaux differentiable at 0, $\|x\| < 1/2$ and $\alpha > 3$. Furthermore, the discussion above concerning the second coordinate of the farthest point tells us that $\phi(q) \leq 1$ implies $(q,\phi(q)) \in L$.

We claim that for all $q \in \frac{1}{2}B_X$

$$\alpha - (r^p - \|x - q\|^p)^{\frac{1}{p}} \leq \phi(q). \tag{3-3}$$

Indeed, if $q \in X$ and $\|q\| \leq 1$, then the element $(q,\beta) \in Y$ is on the sphere $S(u,r)$ if and only if

$$\beta = \alpha \pm (r^p - \|x - q\|^p)^{\frac{1}{p}};$$

because the set L is contained in $B(u,r)$ we have

$$\alpha - (r^p - \|x - q\|^p)^{\frac{1}{p}} \leq \phi(q)$$

for $\{q : \phi(q) \leq 2 - \psi(z_0)\}$. In addition, we know that $\alpha > 3$ so if $\|q\| \leq \frac{1}{2}$ and $|||u - (q,\beta)||| = r$ then $|\alpha - \beta|^p + \|x - q\|^p = r^p$ and thus $|\alpha - \beta|^p \geq r^p - 1 \geq \alpha^p - 1$. If $\beta < \alpha$ then $\beta \leq 1$. Indeed, observe that for $\alpha > \beta > 1$, $(\alpha - \beta)^p < (\alpha - \beta)\alpha^{p-1} <$

$\alpha^p - 1$. However, for all $q \in X$, $(q, \phi(q)) \notin L$ we have $\phi(q) > 1$. Combining these results proves the claim.

Let $g = |||\cdot|||'(-u)$. Observe that if $w = u + v \in B(u,r)$ then $g(w) = g(u) + g(v) = -r + g(v) < 0$. Note that g can be written $g = (g_1, g_2)$ where $g_1 \in X^*$ and $g_2 \in \mathbb{R}$; $(0,\alpha) \in B(u,r)$ so $g((0,\alpha)) < 0$ implies $g_2 < 0$. If G is the support of g and $H = g^{-1}(0)$ then $\{(z,0) : \text{supp} z \in \Gamma \backslash G\}$ is a subspace of H because $g((z,0)) = g_1(z) + g_2 \cdot 0 = 0$. Hence the space $Z = \ell_p(\Gamma \backslash G) \subset X$ is isometrically embedded in H.

Fact 1 in conjunction with the result that $\ell_p(\Gamma)$ has modulus of smoothness of power type p shows that if $h = (y,0) \in H$ with $y \in Z$ then we have

$$|||th - u||| \geq r + K|||th|||^p. \tag{3-4}$$

Let $\beta = \alpha - (r^p - \|x - ty\|^p)^{\frac{1}{p}}$ and consider the vector $v = th - u + (0,\beta)$. An easy calculation shows that $|||v||| = r$ and so $0 > g(u+v) = g(th + (0,\beta)) = g_2 \cdot \beta$ from which $\beta \geq 0$ follows. Thus from (3–4)

$$r + K|||th|||^p \leq |||th - u||| = |||v - (0,\beta)||| \leq r + \beta$$

and so by (3–3) and the fact that $|||th||| = \|ty\|$ we have for $\|ty\| \leq \frac{1}{2}$,

$$\phi(ty) \geq \beta \geq K\|ty\|^p. \tag{3-5}$$

On the other hand consider the function $\phi|_Z$. We know that it is p times Gateaux differentiable and that $\phi|_Z(0) = 0$, so writing its Taylor expansion about 0 (where z^n is the n–tuple consisting of all z's) we have

$$\phi|_Z(z) = \sum_{n=1}^{p} \frac{1}{n!}(\phi|_Z)^{(n)}(0)(z^n) + R_p(z)$$

where $\lim_{t\to 0} R_p(tz)/t^p = 0$. From Fact 2 we can find $y \in Z$, $\|y\| = 1$ such that $\sum_{n=1}^{p-1} \frac{1}{n!}(\phi|_Z)^{(n)}(0)((ty)^n) = 0$ for all $t \in \mathbb{R}$. Because $(\phi|_Z)^{(p)}(0)$ is odd we can also

choose y so that $(\phi|_Z)^{(p)}(0)(ty) < 0$ for all $t > 0$ and for K given we can fix $t_0 \in \mathbb{R}$, $0 < t_0 < \frac{1}{2}$ so that for $\{t : |t| < |t_0|\}$, $\phi|_Z(ty)$ is real valued and $R_p(ty) < K\|ty\|^p$. For this choice of t_0 we have for $\{t : 0 < t < t_0\}$

$$\phi(ty) = \phi|_Z(ty) < K\|ty\|^p$$

which contradicts (3–5). This finishes the proof.

As an immediate consequence of this result we obtain the following:

Corollary 3.3. *(Troyanski) For $p > 1$ odd, there is no p times Gateaux differentiable norm on the space $\ell_p(\Gamma)$ for Γ uncountable.*

CHAPTER 4

Smooth Partitions of Unity in Preduals of WCG Spaces

4.1. Introduction

The initial results concerning smooth partitions of unity in infinite dimensional Banach spaces yielded statements for separable spaces [BF]. The first proof of the existence of C^1–smooth partitions of unity for non–separable spaces was given in [W] where it is shown that any Hilbert spaces admits such a partition. This result was far from optimal, for Toruńczyk proved, among other thing, that all Hilbert spaces admit C^∞–smooth partitions of unity. The means of obtaining this theorem was provided via Theorem 2.1 using the identity mapping from $\ell_2(\Gamma)$ into $c_0(\Gamma)$. This theorem allowed Toruńczyk to prove also that any reflexive space which has a LUR norm which is C^k–smooth admits C^k–smooth partitions of unity.

Subsequent efforts to expand the family of spaces admitting smooth partitions of unity have mainly utilized Theorem 2.1. The next large class of spaces to be invited into this family is the class of WCG spaces. In fact, the paper of [GTWZ] showed that all Banach spaces which are subspaces of WCG spaces and are C^k–smooth admit C^k–smooth partitions of unity.

The work of Johnson and Lindenstrauss [JL] showed that the Banach space JL cannot be the subspace of a WCG Banach space (because its dual is WCG while JL is not WCG) and thus the theorem in [GTWZ] does not provide any information on the partitions question for this space. The aim of this chapter is to show that positive results are obtainable for this and other spaces whose duals are WCG. More precisely, we shall demonstrate the validity of the following theorem:

Theorem 4.1. *If X is C^k–smooth and X^* is WCG, then X admits C^k–smooth partitions of unity.*

To prove this theorem, we shall proceed as in the proof in [GTWZ], but certain fundamental alterations to this line of argument are required. The main obstacle in taking this approach is the lack of a PRI on X, forcing us to work in the bidual in the w^*–topology. Fortunately, the w^*–angelicity of the bidual ball will allow us to extract enough information about X to continue. In addition, there will be no weak compact generating set in X so some of the arguments must be carried from X to its dual. These adjustments are the content of the second section of this chapter; the third section is devoted to constructing a homeomorphism and verifying that it satisfies the conditions requied for the application of [T]. We conclude with some implications of Theorem 4.1.

It should be emphasized that under the conditions of the theorem, we are not guaranteed a PRI on X. Indeed, in the paper [P] it is shown that the space mentioned above, JL, has no PRI. The immediate upshot of this observation is that we cannot expect any type of result connecting PRI's and smooth partitions of unity without some additional restrictions because, as we shall show later, JL admits C^∞–smooth partitions. It seems that this is the first example of a space admitting C^k–smooth partitions of unity, for some $k \geq 1$, while not possessing a PRI.

4.2. Lemmata

In this section we shall lay the groundwork for the proof of the theorem. The following three lemmata allow us to break the space X^* into separable pieces, to map each $x \in X$ into a $c_0(\Gamma)$ space, and to choose a set $D \subset X$ so that each $x \in X$ belongs to a subspace generated by a countable subset of D. Throughout this section and the next, X will denote a Banach space whose dual is the closed linear span of a symmetric, convex, weakly compact set $K \subset X^*$. The first lemma

of this section was proved in Section 2.4 and we include the result here.

Lemma 4.1. *Given X and K, let γ be the ordinal corresponding to the cardinality of the smallest dense subset of X^*. Then there exists a long sequence of projections $\{T_\alpha\}_{\alpha\in\Gamma}$ on X^* with $T_0 = 0$ and T_γ the identity operator, and a family of positive reals $\{D_\alpha\}_{\alpha\in\Gamma_\gamma}$ such that:*

i) $\|T_\alpha\| < \infty$ *for all* $\alpha \in \Gamma$;

ii) $T_\alpha T_\beta = T_\beta T_\alpha = T_\alpha$ *for* $\alpha, \beta \in \Gamma$ *with* $\alpha \leq \beta$;

iii) $f \in \overline{\text{sp}}_{\alpha\in\Gamma_\gamma}\{(T_{\alpha+1} - T_\alpha)(f) : \alpha \in \Gamma\}$ *for each* $f \in X^*$;

iv) $(T_{\alpha+1} - T_\alpha)(X^*)$ *is separable for each* $\alpha \in \Gamma_\gamma$

v) $\{\alpha \in \Gamma_\gamma : \|(T_{\alpha+1} - T_\alpha)(x)\| \geq \epsilon(\|T_{\alpha+1}\| + \|T_\alpha\|\}$ *is finite for all* $\epsilon > 0$ *and all* $x \in X$, *and*

vi) $D_\alpha(T_{\alpha+1} - T_\alpha)(K) \subset K$ *for each* $\alpha \in \Gamma_\gamma$.

Lemma 4.2. *If X and K are given, and $\{T_\alpha\}_{\alpha\in\Gamma}$ is given by Lemma 4.1 then for each $\alpha \in \Gamma_\gamma$ there exists $\{y_n^\alpha\}_{n=1}^\infty \subset X$ whose linear span is w^*–dense in $\tau_\alpha^*(X^{**})$ and for each $x \in X$, $x \neq 0$, $x \in \overline{\text{sp}}\{y_n^\alpha : (n,\alpha) \in \mathbb{N} \times \Lambda(x)\}$.*

Proof: By Lemma 4.1 iv)*, $\tau_\alpha^*(X^{**})$ is w^*–separable so there exists a countable collection of elements $\{\phi_n\}_{n=1}^\infty \subset \tau_\alpha^*(X^{**})$, w^*–dense in the unit ball of $\tau_\alpha^*(X^{**})$. The unit ball of X is w^*–dense in $B_{X^{**}}$ and from Theorem 2.2 we know that the unit ball of X^{**} is Eberlein compact and consequently $w*$–angelic. Because of this, we can find for each $n \in \mathbb{N}$ a sequence $\{x_{in}\}_{i=1}^\infty \subset X$ converging to ϕ_n in the w^* topology of X^{**}, and thus the collection $\{x_{ij} : i, j \in \mathbb{N}\}$ is w^*–dense in the unit ball of $\tau_\alpha^*(X^{**})$. If we rename these elements $\{y_n^\alpha\}_{n=1}^\infty$ then $\overline{\text{sp}}^{w^*}\{y_n^\alpha\}_{n=1}^\infty \supset \tau_\alpha^*(X^{**})$. Now each $\phi \in X^{**}$ satisfies (by iii)*)

$$\phi \in \overline{\bigcup_{\alpha\in\Lambda^*(\phi)} \tau_\alpha^*(X^{**})}^{w^*} \subset \overline{\text{sp}}^{w^*}\{y_n^\alpha : (n,\alpha) \in \mathbb{N} \times \Lambda^*(\phi)\},$$

so in particular $x \in \overline{\text{sp}}^{w^*}\{y_n^\alpha : (n,\alpha) \in \mathbb{N} \times \Lambda^*(x)\}$ and $\{x\} \cup \{y_n^\alpha : (n,\alpha) \in \mathbb{N} \times \Lambda^*(x)\} \subset X$. But if x is the w^* limit of elements in X then it is the weak limit of the same elements, or in other words, $x \in \overline{\text{sp}}^{w}\{y_n^\alpha : (n,\alpha) \in \mathbb{N} \times \Lambda^*(x)\}$. Since this is a convex set, we get $x \in \overline{\text{sp}}\{y_n^\alpha : (n,\alpha) \in \mathbb{N} \times \Lambda^*(x)\}$ by Mazur's theorem.

Lemma 4.3. *Let X, K, $\{T_\alpha\}_{\alpha\in\Gamma}$ and $\{D_\alpha\}_{\alpha\in\Gamma_\gamma}$ be as in Lemma 4.1. Define a norm $|||\cdot|||$ on X^{**} by $|||\phi||| = \sup_{f\in K}|\phi(f)|$ for $\phi \in X^{**}$. Then for each $x \in X$*

$$\left(D_\alpha |||\tau_\alpha^*(x)|||\right)_{\alpha\in\Gamma_\gamma} \in c_0(\Gamma_\gamma).$$

Proof: We wish to prove this lemma by contradiction so we suppose it is not true. Then there exists $x \in X$, $\epsilon > 0$ and $A \subset \Lambda^*(x)$, $|A|$ infinite, such that $|||D_\alpha\tau_\alpha^*(x)||| > \epsilon$ for each $\alpha \in A$. This means we can find for each $\alpha \in A$ some $f'_\alpha \in K$ such that $D_\alpha\big(\tau_\alpha^*(x)\big)(f'_\alpha) > \epsilon$ or equivalently $D_\alpha \cdot \big[\tau_\alpha(f'_\alpha)\big](x) > \epsilon$. If we define $f_\alpha = D_\alpha\tau_\alpha(f'_\alpha)$ then $f_\alpha \in K$ for each $\alpha \in A$, so there exists $f \in K$ a weak limit point of f_α. By the Eberlein–Šmulian theorem we obtain $f_{\alpha_n} \xrightarrow{w} f$, with α_n distinct elements of A so, in particular, $\tau_\alpha^*(\phi)(f) = \lim_{n\to\infty}\tau_\alpha^*(\phi)(f_{\alpha_n})$ for each $\alpha \in \Gamma_\gamma$ and each $\phi \in X^{**}$. Noting that $\phi(f_{\alpha_n}) = 0$ for all ϕ with $\tau_{\alpha_n}^*(\phi) = 0$ we conclude that $\tau_\alpha^*(\phi)(f) = 0$ for each $\alpha \in \Gamma_\gamma$ and each $\phi \in X^{**}$. Using property iii)* of the PRI $\{T_\alpha\}_{\alpha\in\Gamma}$, $\phi \in \overline{\text{sp}}^{w^*}\{\tau_\alpha^*(\phi) : \alpha \in \Gamma_\gamma\}$ so that $\phi(f) = 0$ for each $\phi \in X^{**}$ and so $f = 0$. But $f_{\alpha_n}(x) > \epsilon$ contradicts $f_{\alpha_n}(x) \to f(x) = 0$ so the lemma is true.

4.3. Proof of Theorem 4.1

Having completed the proofs of the lemmata, we can now commence with the proof of the theorem. Our strategy will be to use Theorem 2.1 by creating an index set Δ, exhibiting a map $h : X \to c_0(\Delta)$ and then verifying that h is an embedding as required. In order to construct such a map we first develop some concepts which

will be useful in this construction and as a consequence we begin the description of the index set Δ.

We begin by fixing $\{T_\alpha\}_{\alpha\in\Gamma}$ as in Lemma 4.1 and choosing $\{y_n^\alpha : (n,\alpha) \in \mathbb{N}\times\Gamma_\gamma\}$ by Lemma 4.2. Note that because $\tau_\alpha(K)$ is linearly dense in the separable space $\tau_\alpha(X^*)$ we can choose, for each $\alpha\in\Gamma_\gamma$, a sequence

$$\{f_n^\alpha\}_{n=1}^\infty \subset D_\alpha\tau_\alpha(K) \qquad (4-1)$$

such that the f_n^α are separating for $\tau_\alpha^*(X^{**})$. Next, by Theorem 2.2, we find a linear 1–1 continuous map $E : X^{**} \to c_0(\Theta)$ for some set Θ.

Because X is C^k–smooth, there is a function $\mu \in C^k(X,\mathbb{R})$ such that $\sup\mu(x) = M \neq 0$ and $\mathrm{supp}\mu$ is bounded. By composing μ, if necessary, with a function $\zeta \in C^\infty(\mathbb{R},\mathbb{R})$ satisfying $\zeta(t) = 0$ if $t < 0$ $\zeta(t) = t$ if $t > M/2$ and $0 < \zeta(t) < M$ for all $t \in \mathbb{R}$, we can assume that μ is everywhere positive. Let $x_0 \in X$ be such that $\mu(x_0) > 7M/8$ and find $N \in \mathbb{N}$ such that $\|x\| \geq N$ implies $\mu(x_0 - x) = 0$. We define $\nu(x) = \frac{1}{M}\mu(x_0 - Nx)$. Then $\nu \in C^k(X,\mathbb{R})$ and

i) $0 \leq \nu(x) \leq 1$ for all $x \in X$

ii) ν is zero outside the unit ball of X

iii) $\nu(0) > 7/8$.

By continuity we may fix a real number r, $0 < r < 1$, such that $\nu(x) > 3/4$ whenever $\|x\| < r$. Now find a function $\psi \in C^\infty(\mathbb{R},[0,1])$ with $\psi(x) = 0$ for $|x| \geq 3/4$ and $\psi(x) = 1$ for $|x| \leq 5/8$. If we set $\Psi_n(x) = \psi\circ\nu(nx)$, then $0 \leq \Psi_n(x) \leq 1$ and

$$\Psi_n(x) = \begin{cases} 0 & \|x\| < r/n \\ 1 & \|x\| \geq 1/n. \end{cases}$$

We are in position to define a family of projections on X; in doing so we continue the construction of Δ. To this end define $F = \{(a,b,c) \in Q^n \times \mathbb{N}^n \times \Gamma_\gamma^n, 1 \leq n < \infty\}$ and enumerate $\bigcup_{n=1}^\infty Q^n = \{\rho_i\}_{i=1}^\infty$ and $\bigcup_{n=1}^\infty \mathbb{N}^n = \{\eta_i\}_{i=1}^\infty$. Now if

$\rho_i = (r_{i_1}, \ldots, r_{i_m}) \in \mathbb{Q}^m$, $\eta_j = (n_{j_1}, \ldots, n_{j_m}) \in \mathbb{N}^m$ and $c = (\alpha_1, \ldots, \alpha_m) \in \Gamma_\gamma^m$ we let $P_{\rho_i \eta_j c}$ be a norm one projection of X onto the one dimensional subspace $\operatorname{sp} \sum_{k=1}^m r_{i_k} y_{n_{j_k}}^{\alpha_k}$. In connection with this process of indexation it will be important to avoid repetitions in an m-tuple $c = (\alpha_1, \ldots, \alpha_m) \in \Gamma_\gamma^m$ so we define $b(c)$ to be the distinct elements of the set $\{\alpha_1, \ldots, \alpha_m\}$. The need for such a distinction will become clear as the proof develops.

If $S(\Gamma)$ denotes the finite subsets of Γ_γ and $\mathbb{N}'$ is a copy of $\mathbb{N}$, then we define Δ to be the disjoint union of $\mathbb{N}$, $\mathbb{N}'$, Θ, $S(\Gamma)$ and $F \times \mathbb{N}$. We now define our map h from X into $c_0(\Delta)$ coordinate-wise in $\delta \in \Delta$;

$$h_\delta(x) = \begin{cases} \frac{1}{n} \nu(\frac{1}{n} x) & \delta = n \in \mathbb{N} \\ \frac{1}{n'} \Psi_{n'}(x) & \delta = n' \in \mathbb{N}' \\ E_\theta(x) & \delta = \theta \in \Theta \\ \prod_{\alpha \in b} \Big(\sum_{m=1}^\infty \frac{1}{2^m} \big(f_n^\alpha(x)\big)^2 \Big) & \delta = b \in S(\Gamma) \\ \frac{1}{ijk} \prod_{\alpha \in b(c)} \Big(\sum_{m=1}^\infty \frac{1}{2^m} \big(f_n^\alpha(x)\big)^2 \Big) \Psi_k\big[(I - P_{\rho_i \eta_j c})(x)\big] & \\ & \delta = \big((\rho_i, \eta_j, c), k\big) \in F \times \mathbb{N}. \end{cases}$$

To see that h maps X into $c_0(\Delta)$ we fix $\epsilon > 0$ and $x \in X$. If $D \subset \Delta$ we shall say that $|h(x)| \geq \epsilon$ finitely often in D if there exists only finitely many $\delta \in D$ with $|h_\delta(x)| \geq \epsilon$. We shall show that $|h(x)| \geq \epsilon$ finitely often in each of $\mathbb{N}$, $\mathbb{N}'$, Θ, $S(\Gamma)$ and $F \times \mathbb{N}$.

If $n \in \mathbb{N}$ or $\mathbb{N}'$ satisfies $n > \epsilon^{-1}$ then $|h_n(x)| < \epsilon$ and since E maps X into $c_0(\Theta)$, $|h_n(x)| \geq \epsilon$ finitely often in each of $\mathbb{N}$, $\mathbb{N}'$ and Θ. Turning to indices in $S(\Gamma)$, we note first that by (4–1) $f_n^\alpha \in D_\alpha \tau_\alpha(K)$ for each $n \in \mathbb{N}$ and $\alpha \in \Gamma_\gamma$ so there exists $g_n^\alpha \in K$ such that $f_n^\alpha = D_\alpha \tau_\alpha(g_n^\alpha)$. It follows from the definition of $|||\cdot|||$

(see Lemma 4.3) that

$$(f_n^\alpha(x))^2 = |D_\alpha \tau_\alpha^*(x) g_n^\alpha|^2 \leq |||D_\alpha \tau_\alpha^*(x)|||^2$$

hence $\sum_{n=1}^{\infty} \frac{1}{2^n}(f_n^\alpha(x))^2 \leq |||D_\alpha \tau_\alpha^*(x)|||^2$. Lemma 4.3 assures us that only finitely many choices of $\alpha \in \Gamma_\gamma$, say N of them, satisfy $|||D_\alpha \tau_\alpha^*(x)||| \geq 1$ so by setting $M = \max_{\alpha \in \Gamma_\gamma} |||D_\alpha \tau_\alpha^*(x)|||$ we obtain, for all $c \in S(\Gamma)$,

$$\prod_{\alpha \in c} \Big(\sum_{n=1}^{\infty} \frac{1}{2^n} (f_n^\alpha(x))^2 \Big) \leq \prod_{\alpha \in c} |||D_\alpha \tau_\alpha^*(x)|||^2 \leq M^{2N}.$$

If $b = (\beta_1, \ldots, \beta_k) \in S(\Gamma)$ is such that for some j, $1 \leq j \leq k$, $|||N(\beta_j)\tau_{\beta_j}^*(x)||| < \sqrt{\epsilon}/M^N$ then

$$\prod_{\alpha \in b} \Big(\sum_{n=1}^{\infty} \frac{1}{2^n} (f_n^\alpha(x))^2 \Big) < \Big(\prod_{\substack{\alpha \in b \\ \alpha \neq \beta_j}} |||D_\alpha \tau_\alpha^*(x)|||^2 \Big) \frac{\epsilon}{M^{2N}} \leq \epsilon.$$

Again by Lemma 4.3, only finitely many $b \in S(\Gamma)$ are not of this form so $|h(x)| \geq \epsilon$ finitely often in $S(\Gamma)$.

To finish, we examine indices of the form $\delta = [(\rho_i, \eta_j, c), k] \in F \times \mathbb{N}$. Observe that $\Psi_k[(I - P_{\rho_i \eta_j c})(x)] \leq 1$ so if i, j or k are more than ϵ/M^{2N} then $|h_\delta(x)| < \epsilon$. But for fixed i, $\rho_i \in Q^m$ for some $m \in \mathbb{N}$ so the element c must be in Γ_γ^m. In particular, the preceeding paragraph showed that there exists a finite subset of $B \subset \Gamma_\gamma$ such that if $a \in S(\Gamma)$ and $a \not\subset B$ then $h_a(x) < \epsilon$. The use of the notation $b(c)$ is now justified, for if m is fixed, there are only finitely many choices of $c \in \Gamma_\gamma^m$ for which all coordinates belong to B. The notation excludes any possibility of duplication of terms so we can use the result of the preceding paragraph to deduce that for fixed i,

$$\prod_{\alpha \in b(c)} \Big(\sum_{n=1}^{\infty} \frac{1}{2^n} (f_n^\alpha(x))^2 \Big) \geq \epsilon$$

for only a finite number of choices of c. Thus $|h(x)| \geq \epsilon$ finitely often in $F \times \mathbb{N}$ so we see indeed that h maps X into $c_0(\Delta)$.

By Lemma 2.2 the map h is continuous and it is 1–1 because E is. To see that the components of h are C^k–times differentiable, first note that each component involves the compositions of functions each of which is at least k times differentiable (note that we require the functions here to be Fréchet differentiable). The differentiability of the sums appearing in the fourth and fifth lines in the definition of h is guaranteed because each such sum is uniformly convergent. This is true because the functions f_n^α are linear and bounded uniformly by one, thus the k^{th} derivative of the sum is bounded by 2^k; the linearity ensures infinite differentiability.

The last condition required by [T] is that $h^{-1} : h(X) \to X$ be continuous. Because h is 1–1, we can use Lemma 2.1. Fix $\{x_k\}_{k=1}^\infty$ and x such that $h(x_k) \to h(x)$ We claim that $\{x_k\}_{k=1}^\infty \cup \{x\}$ is bounded. Indeed, for some $n \in \mathbb{N}$, $h_n(x) = 1/n\nu(x/n) \neq 0$ and by assumption $h_n(x_k) \to h_n(x) \neq 0$. Thus for only finitely many k does $h_n(x_k) = 0$; by the definition of ν, the remaining x_k live inside the n-ball of X so our claim is valid.

Finally we show that given any $\epsilon > 0$ there exists a one dimensional subspace $Y \subset X$ and $N \in \mathbb{N}$ such that $\inf_{y \in Y} \|x_k - y\| < \epsilon$ for all $k \geq N$. Fix $\epsilon > 0$. If $x = 0$ then $h_{n'}(x_k) \to h_{n'}(x) = 0$ for each $n' \in \mathbb{N}'$, so $x_k \to 0$ and we can choose Y to be arbitrary. If $x \neq 0$ then by Lemma 4.2, $x \in \overline{\text{sp}}\{y_n^\alpha : (n, \alpha) \in \mathbb{N} \times \Lambda^*(x)\}$ so we can find $n_0 > \epsilon^{-1}$ and $(\rho_i, \eta_j, c) \in F$ such that $\|(I - P_{\rho_i \eta_j c})(x)\| < r/n_0$ and $c = (\alpha_1, \ldots, \alpha_m) \in [\Lambda^*(x)]^m$. Because $\Psi_{n_0}(t) = 0$ for $|t| \leq r/n_0$ we have $h_{[(\rho_i, \eta_j, c), n_0]}(x) = 0$. In addition, for each $\alpha \in \Gamma_\gamma$ the sequences $\{f_n^\alpha\}_{n=1}^\infty$ are separating on $\tau_\alpha^*(X^{**})$ so $\sum_{n=1}^{\infty} \frac{1}{2^n}(f_n^\alpha(x))^2 \neq 0$ for each $\alpha \in \Lambda^*(x)$. In particular $\prod_{\alpha \in b(c)} \sum_{n=1}^{\infty} \frac{1}{2^n}(f_n^\alpha)^2(x) \neq 0$ so $h_{b(c)}(x) = 2\lambda$ for some constant $\lambda > 0$. From these

observations it follows that there exists an integer N such that for all $k > N$,

$$h_{[(\rho_i,\eta_j,c),n_0]}(x_k) < \lambda/(2ijn_0) \qquad \text{and} \qquad h_{b(c)}(x_k) > \lambda, \tag{4-2}$$

because of the coordinate–wise convergence of $h(x)$. Note that $h_{[(\rho_i,\eta_j,c),n_0]}(x_k) = \frac{1}{ijn_0} h_{b(c)}(x_k)\Psi_{n_0}\big((I - P_{\rho_i\eta_j c})(x_k)\big)$, so (4–2) forces $\Psi_{n_0}\big((I - P_{\rho_i\eta_j c})(x_k)\big) < 1/2$ which implies $\|(I - P_{\rho_i\eta_j c})(x_k)\| < 1/n_0 < \epsilon$ for all $k > N$. Thus we have shown, for all $k > N$, that

$$\inf_{y \in P_{\rho_i\eta_j c}(X)} \|x_k - y\| < \epsilon.$$

It follows that $\{x_k\}_{k=1}^{\infty} \cup \{x\}$ is totally bounded, completing the proof of the theorem.

4.4 Corollaries

Combining the main theorem of this chapter and Theorem 3.1 in [GTWZ] we have the following direct result.

Corollary 4.1. *If a Banach space X is C^k–smooth and if either X or X^* is WCG then X admits C^k–smooth partitions of unity.*

As mentioned earlier, we shall be primarily interested in spaces for which the dual space admits a PRI. The work of Godefroy *et. al.* indicates that any space we consider should not be the subspace of a WCG space. The Banach space which forms the motivation and the justification for the theorem in Chapter 4 is defined here and its salient properties are listed. The development of this theory is taken from the unpublished lecture notes of Amir [A].

To begin, we observe that there is an uncountable family $\{N_\gamma : \gamma \in \Gamma\}$ of infinite subsets of $\mathbb{N}$ such that $N_{\gamma_1} \cap N_{\gamma_2}$ is finite for $\gamma_1 \neq \gamma_2$. To see that this is true, identify $\mathbb{N}$ with the rationals and let Γ be the irrational numbers. For each γ, take a sequence of rationals converging to γ and let N_γ be the corresponding set in

$\mathbb{N}$. With this family of subsets chosen we let $\ell_\infty(\mathbb{N})$ denote the set of all bounded functions on $\mathbb{N}$ and we let $x_\gamma \in \ell_\infty(\mathbb{N})$ be the characteristic function of N_γ. Define

$$E = \mathrm{sp}\{x_\gamma : \gamma \in \Gamma\} + c_0(\mathbb{N}) = \{z \in \ell_\infty(\mathbb{N}) : z = \sum_{i=1}^{n} a_{\gamma_i} x_{\gamma_i} + y,\ y \in c_0(\mathbb{N})\}$$

and equip E with a norm $\|\cdot\|$ given by

$$\|\sum_{i=1}^{n} a_{\gamma_i} x_{\gamma_i} + y\| = \max(\|\sum_{i=1}^{n} a_{\gamma_i} x_{\gamma_i} + y\|_\infty, (\sum_{i=1}^{n} |a_{\gamma_i}|^2)^{1/2}).$$

The space JL is the closure of E with respect to this norm and has the following properties:

1) $c_0(\mathbb{N})$ is a closed subspace of JL

2) the quotient of JL modulo $c_0(\mathbb{N})$ is isomorphic to a Hilbert space

3) JL^* is WCG

4) JL is not a subspace of a WCG space

5) JL is C^∞–smooth.

Statements 1)–4) are proved in [JL] while 5) follows from the results in [GPWZ]. Statements 3) and 5) combined with Theorem 4.1 immediately gives the following corollary.

Corollary 4.2. *The space JL has C^∞–smooth partitions of unity.*

The next set of examples shows an interesting pattern associated with the higher dual spaces of the famous space JT (also known as the James Tree space). Created by R.C. James, this space has been the source of many counter–examples in Banach space theory and its family of conjugates provides a striking result concerning partitions of unity. Before recalling a theorem from [LS] we agree to write JT^n to denote the n^{th} dual of JT.

Theorem 4.2. *(Lindenstrauss–Stegall) JT is separable, JT^* is not separable and JT^n is WCG precisely when n is even.*

Remarks: 1) It is immediate from [GTWZ] that JT^n is C^1–smooth for n odd. Indeed, JT^{2n} admit a dual LUR norm whenever $n \geq 1$ and thus by the well–known result of Lovaglia (cf. e.g. [D]) JT^{2n-1} admits a C^1–smooth norm.
2) JT^{2n} is not C^1–smooth for any $n \geq 0$. To see this, note that JT itself is separable while its dual is not. This implies immediately that JT has no C^1–smooth function with nonempty bounded support (cf. e.g. [DGZ$_2$]) and so JT is not C^1–smooth. Since JT is a closed subspace of each of its higher even duals, none of these spaces is C^1–smooth.

Combining the above results we have:

Corollary 4.3. *JT^n admits C^1–smooth partitions of unity if and only if $n \in \mathbb{N}$ is odd.*

It should be noted that this result can be obtained also from the work of [V].

We close this chapter with a question. The paper [GTWZ] shows that being a C^k–smooth subspace of a WCG space is sufficient to obtain C^k–smooth partitions of unity but the problem seems more difficult in the case where the dual is a subspace of a WCG space. Thus we ask: is it true that Theorem 4.1 can be extended to the case where the dual is a subspace of a WCG space?

CHAPTER 5

Convergence of Norms and Smooth Partitions of Unity

1. Introduction

In the previous chapter we used the existence of a PRI on the dual to find a suitable decomposition on the original space. This was made possible by finding an injection into $c_0(\Gamma)$ via the bidual and using the existence of a generating weak compact set on the dual. This chapter will be devoted to easing the conditions on the dual space by removing the existence of a generating weak compact set. Naturally, we would like to ensure that certain aspects of the previous method will still be available, and for this we shall supply the base space with a WLUR norm which is the limit of differentiable norms (of a given degree) uniformly on bounded sets.

To motivate this approach, let us recall the result in [T] which states that a reflexive Banach space admits C^k–smooth partitions of unity whenever it admits a LUR norm which is C^k–smooth away from the origin. An attempt to generalize this theorem must deal with the following observation; Asplund's averaging technique (cf. [D]) for higher order smooth norms is in general not available (e.g $c_0(\mathbb{N})$ has a C^∞–smooth norm and its dual has a LUR norm, but $c_0(\mathbb{N})$ has no LUR C^2–smooth norm [FWZ]). The theorem we prove in this chapter addresses this observation. Because the existence of a LUR C^k–smooth norm is too restrictive, we shall require only that a space has a WLUR norm which is the limit of C^{k+1}–smooth norms, uniformly on bounded sets. As we shall see, this theorem does not generalize the result of Toruńczyk quoted above, but does yield a similar statement.

It should be noted that we do not assume *a priori* that the space in question admits a mapping into some $c_0(\Gamma)$.

Theorem 5.1. *Suppose a WLUR norm on a Banach space X can be uniformly approximated on bounded sets by equivalent C^{k+1}–smooth norms, where $k \in \mathbb{N} \cup \{\infty\}$. Then X admits C^k–smooth partitions of unity.*

The proof of this theorem appears in the following section while consequences and concrete examples satisfying these conditions will be elicited in Section 5.3.

2. Proof of Theorem 5.1

The proof of the theorem in question is quite involved and so it is natural to structure the proof so as to be able both to follow the line of the arguments comprising the proof and to examine the purpose of the plethora of notations and constructs which will arise. The essential idea is to proceed as in the preceding chapter by producing a homeomorphic embedding and applying Toruńczyk's theorem. There are many similarities in the proof of this theorem and that of Chapter 4, but of course the differences are substantial, and therefore warrant individual attention. The crucial difference is the means used to designate a convenient collection of elements from which we can obtain convergence properties for the homeomorphism which is eventually given. In Theorem 4.1, w^*–angelicity of the bidual unit ball in conjunction with PRI is used to find such a collection; here, the convexity property, the convergence of norms, and the PRI are combined to provide it. In addition, no 1–1 map into $c_0(\Gamma)$ in available *a priori* in our situation. Such a map is given only on the dual and thus additional work is required to ensure the 1–1 nature of the constructed map.

Our plan is essentially simple: we shall go into the dual to obtain a PRI, find a 1–1 map of the dual into some c_0 space, collect families of elements in X which will yield a sufficient degree of density, and then define a homeomorphism via the derivatives of the given norms. We shall break the proof into steps as follows; Step

1 introduces the notation, Step 2 contains two preparatory lemmata, and the final step defines a homeomorphism and proves that it has the properties required to yield the theorem. We proceed with the proof.

Step 1. In this portion of the proof, we shall establish some conventions and introduce the bulk of the notation required. We let X be a Banach space such that X has a WLUR norm $\|\cdot\|$. Since there are norms which converge uniformly on bounded sets to $\|\cdot\|$, we can suppose, choosing a subsequence if necessary, that $\{\|\cdot\|_j\}_{j=1}^{\infty}$ is a sequence of C^{k+1}–smooth norms on X such that for all $x \in X$ with $\|x\| \leq 2$ we have

$$\big|\|x\|_j - \|x\|\big| < \frac{1}{j}. \tag{5-1}$$

In particular X has a C^1–smooth norm, so X is an Asplund space and by [FG] we have a projectional resolution of the identity on X^*. We recall from Section 2.2 that X^* admits a separable PRI $\{T_\alpha\}_{\alpha\in\Gamma}$ (γ the first ordinal of cardinality $\operatorname{dens} X$). We let S_γ denote the set of all finite subsets of Γ_γ and for each $F \in S_\gamma$ let $\{f_i^F\}_{i=1}^{\infty}$ be a dense sequence in $\overline{\operatorname{sp}}_{\alpha\in F}\tau_\alpha(X^*)$; then for each triple F, i, j there exist a sequence $\{x_{ijk}^F\}_{k=1}^{\infty}$ such that $\|x_{ijk}^F\|_j = 1$ and $\lim_{k\to\infty} f_i^F(x_{ijk}^F) = \|f_i^F\|_j$.

In order to use the technique of Kadets, we wish to have a collection of indexed one dimensional subspaces of X which, collectively, are dense in X. In this direction we define a family of norm one projections (with respect to $\|\cdot\|$) onto the spans of rational linear combinations of elements of the form x_{ijk}^F selected above. First, we enumerate $\bigcup_{n=1}^{\infty}\mathbb{Q}^n = \{\rho_i\}_{i=1}^{\infty}$ and $\bigcup_{n=1}^{\infty}\mathbb{N}^n = \{\eta_i\}_{i=1}^{\infty}$, and let S_γ^m denote the m–fold product of S_γ. Now let

$$S = \{(a, \mathcal{F}, b, c, d) \in \mathbb{Q}^n \times S_\gamma^n \times \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^n, 1 \leq n < \infty\};$$

if $s = (\rho_m, \mathcal{F}, \eta_i, \eta_j, \eta_k) \in S$ with $\rho_m = (r_{m_1}, \ldots, r_{m_n})$, $\mathcal{F} = (F_1, \ldots, F_n)$, $\eta_i = (i_1, \ldots, i_n)$, $\eta_j = (j_1, \ldots, j_n)$, and $\eta_k = (k_1, \ldots, k_n)$ then we set $o(s) = n$ if $\rho_m \in \mathbb{Q}^n$

and we define π_s to be the norm one projection of X onto the one-dimensional subspace $\mathrm{sp}\sum_{l=1}^{o(s)} r_{m_l} x^{F_l}_{i_l j_l k_l}$.

Step 2. The lemmata which are to be shown here state that under the conditions of the theorem, the subspaces introduced in the previous step are indeed dense (in the aggregate) and that it is possible to find a "nice" mapping from X^* into $c_0(\Theta)$ for some Θ. Before attacking these problems, we make the following observation concerning equation $(5-1)$.

The inequality $(5-1)$ make an assertion concerning elements which are bounded with respect to $\|\cdot\|$ but we shall have occasion to demand that knowledge of x with respect to $\|\cdot\|_l$ will yield similar information of x with respect to $\|\cdot\|$. More precisely, if $\|x\|_l = 1$ what can be said of $\|x\|$? We claim that $\|x\| \leq 1 + 1/l$. To see this, let $\|x\|_l = 1$ and suppose $\|x\| = M$. By appealing to equation $(5-1)$ and using homogeneity of $\|\cdot\|$ we have

$$\left|\|x\| - \|x\|_l\right| < \frac{M}{2l}.$$

We can assume $M \geq 1$ so we obtain $M - 1 < M/2l$ or equivalently, $M < 2l/(2l-1) \leq 1 + 1/l$ and the claim is validated. We are now in a position to move to the lemmata.

Lemma 5.1. *Under the assumption of Theorem 5.1 and with the notation developed in Step 1, for given $\epsilon > 0$ there exists $s = (\rho_m, \mathcal{F}, \eta_i, \eta_j, \eta_k) \in S$ with $F_l \subset \{\alpha : \tau_\alpha[\|\cdot\|'_{j_l}(x)] \neq 0\}$ for each $l \leq o(s)$ such that $\|x - \pi_s(x)\| < \epsilon$.*

Proof: Without loss of generality we can and do assume that $\|x\| = 1$ and we fix $\epsilon > 0$. To begin, we wish to find a finite subset of Γ_γ such that $\|\cdot\|'_j$ is essentially supported on this subset; by this we mean that we would like to find, for each j, a finite set $F_j \subset \Lambda(\|\cdot\|'_j(x))$ such that there exists $f \in \mathrm{sp}\cup_{\alpha\in F_j} \tau_\alpha(X^*)$ with $\|f\|_j = 1$ and $\left\|f - \|\cdot\|'_j(x)\right\|_j < j^{-1}$. The existence of the PRI $\{T_\alpha\}_{\alpha\in\Gamma}$ (in

particular property iii)) states that this is always possible and so we fix these sets F_j. Next find i_j such that $\|f_{i_j}^{F_j} - f\|_j < j^{-1}$ and since this implies $\|f_{i_j}^{F_j}\|_j > 1 - j^{-1}$ we can find k_j so that $f_{i_j}^{F_j}(x_{i_j j k_j}^{F_j}) > 1 - 2j^{-1}$.

Now note that for each $j \in \mathbb{N}$, $\|x + x_{i_j j k_j}^{F_j}\| \leq \|x\| + \|x_{i_j j k_j}^{F_j}\| \leq 2 + j^{-1}$ and since $\|x\| = 1$ implies $\|x\|_j \leq 1 + 1/j$, it follows from the observation above that $\|x + x_{i_j j k_j}^{F_j}\|_j \leq (1 + 1/j)(1 + 1/2j)\|x + x_{i_j j k_j}^{F_j}\| \leq (1 + 2/j)\|x + x_{i_j j k_j}^{F_j}\| \leq \|x + x_{i_j j k_j}^{F_j}\| + 6/j$. If we combine these results we have

$$\begin{aligned}
\|x + x_{i_j j k_j}^{F_j}\| + 6j^{-1} &\geq \|x + x_{i_j j k_j}^{F_j}\|_j \geq f(x + x_{i_j j k_j}^{F_j}) \\
&= f_{i_j}^{F_j}(x_{i_j j k_j}^{F_j}) - (f_{i_j}^{F_j} - f)(x_{i_j j k_j}^{F_j}) + [\|\cdot\|_j'(x)](x) \\
&\quad - (\|\cdot\|_j'(x) - f)(x) \\
&\geq 1 - 2j^{-1} - \|f_{i_j}^{F_j} - f\|_j + \|x\|_j\left(1 - \left\|\|\cdot\|_j'(x) - f\right\|_j\right) \\
&\geq 1 - 2j^{-1} - j^{-1} + (1 - j^{-1})(1 - j^{-1}) \geq 2 - 5j^{-1}.
\end{aligned}$$

We now conclude that $\left|\|x + x_{i_j j k_j}^{F_j}\| - 2\right| < 11n^{-1}$; it follows by the WLUR of $\|\cdot\|$ that $x_{i_j j k_j}^{F_j} \xrightarrow{w} x$, hence there exists a sequence of convex combinations of these elements which converge in norm to x. Find such a convex combination $y = \sum_{m=l_1}^{l_n} b_m x_{i_m m k_m}^{F_m}$, $b_m \in \mathbb{R}$, such that $\|y - x\| < \epsilon/2$. Now for each b_m find a rational number r_m such that $|b_m - r_m| < \epsilon/2n$. Thus we have the estimate $\|x - \sum_{m=l_1}^{l_n} r_m x_{i_m m k_m}^{F_m}\| \leq \|x - y\| + \|y - \sum_{m=l_1}^{l_n} r_m x_{i_m m k_m}^{F_m}\| \leq \epsilon/2 + n(\epsilon/2n) = \epsilon$ which yields the desired result.

Lemma 5.2. *There exists a linear, 1-1 operator $T : X^* \to c_0(\Gamma_\gamma \times \mathbb{N})$ with $\|T\| \leq 1$, where γ is the first ordinal of cardinality for dens(X). Furthermore if $\alpha \in \Gamma_\gamma$ and $\tau_\alpha(f) \neq 0$, then for some $n \in \mathbb{N}$ we have $T_{(\alpha,n)} f \neq 0$.*

Proof: In this proof we shall work exclusively in the norm $\|\cdot\|$ and we write S_{X^*} and $S_{X^{**}}$ for the dual and bidual spheres of X with respect to this norm. We

first choose, for each $\alpha \in \Gamma_\gamma$, a sequence $\{g_n^\alpha\}_{n=1}^\infty$ dense in $S_{X^*} \cap \tau_\alpha(X^*)$ and a sequence $\{\phi_n^\alpha\}_{n=1}^\infty \subset S_{X^{**}}$ such that $\phi_n^\alpha(g_n^\alpha) = 1$. The operator T is then defined at the coordinate (α, n) by

$$T_{(\alpha,n)}(f) = \frac{\phi_n^\alpha(\tau_\alpha(f))}{n(\|T_{\alpha+1}\| + \|T_\alpha\|)}$$

for each $f \in X^*$. By property v) of the PRI, $\{\alpha \in \Gamma_\gamma : \|\tau_\alpha(f)\| > \epsilon(\|T_{\alpha+1}\| + \|T_\alpha\|)\}$ is finite for each $\epsilon > 0$ and each $f \in X^*$, so T maps into $c_0(\Gamma_\gamma \times \mathbb{N})$. If $\tau_\alpha(f) \neq 0$ (say $\|\tau_\alpha(f)\| = c$) then for some n, $\|\tau_\alpha(f) - cg_n^\alpha\| < c/2$ so $\phi_n^\alpha[\tau_\alpha(f)] > 0$ and $T_{(\alpha,n)}f \neq 0$. The operator T is clearly linear, one–to–one and $\|T\| \leq 1$.

Lemma 5.3. *If $x = (x_\alpha)_{\alpha\in\Gamma_\gamma} \in c_0(\Gamma_\gamma)$ then $y = (\prod_{\alpha\in F} x_\alpha)_{F\in S_\gamma} \in c_0(S_\gamma)$.*

Proof: Fix $\epsilon > 0$ and note that $\max_{\alpha\in\Gamma_\gamma} |x_\alpha| = M < \infty$. Since $x \in c_0(\Gamma_\gamma)$ we know there exists a finite set $A \subset \Gamma_\gamma$ of cardinality n such that for $\alpha \notin A$, $|x_\alpha| < 1$. Now we can find a set $B \subset \Gamma_\gamma$ such that $\beta \notin B$ implies $|x_\beta| < \epsilon/M^n$. Now if $F \in S_\gamma$ and there exists $\sigma \in F$ with $\sigma \notin B$ then

$$\prod_{\alpha\in F} x_\alpha = x_\sigma \Big(\prod_{\substack{\alpha\in F \\ \alpha\neq\sigma}} x_\alpha \Big) < \epsilon.$$

Because only finitely many choices of $F \in S_\gamma$ contain only elements from B, the proof is complete.

Step 3. For each $p \in \mathbb{N}$ choose a C^∞–smooth function $\psi_p : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} &\psi_p(t) = \tfrac{1}{2p} && |t| \leq \tfrac{1}{2p} \\ &\tfrac{1}{2p} \leq \psi_p(t) \leq \tfrac{1}{p} && \tfrac{1}{2p} < |t| < \tfrac{1}{p} \\ &\psi_p(t) = |t| && |t| \geq \tfrac{1}{p}. \end{aligned}$$

We introduce these functions for two reasons. First they are constant on a neighbourhood of the origin and thus when composed with an equivalent norm will retain

the value of the norm outside a small prescribed region yet the composition will be everywhere differentiable to the same degree as the norm. Second, there is no ready–made linear 1–1 injection into $c_0(\Gamma)$ so the introduction of these functions will provide a means of obtaining a 1–1 homeomorphism.

To describe this homeomorphism we let Δ be the disjoint union of $S \times \mathbb{N}^3$, $S_\gamma \times \mathbb{N}^2$, $\Gamma_\gamma \times \mathbb{N}^3$, and $\mathbb{N}^2$, and define $h : X \to c_0(\Delta)$ at the coordinates of Δ and for each $x \in X$ as follows;

$$h_\delta(x) = \begin{cases} \dfrac{\psi_q\|x - \pi_s(x)\|_l}{4mijklqp} \displaystyle\prod_{\substack{\alpha \in F_r \\ 1 \le r \le o(s)}} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1}\left[\left(T_{(\alpha,n)}\{(\psi_p \circ \|\cdot\|_{j_r})'(x)\} \right)^2 \right] \right) \\ \qquad \delta = (s,l,p,q) \in S \times \mathbb{N}^3 \quad s = (\rho_m, \mathcal{F}, \eta_i, \eta_j, \eta_k) \qquad (1) \\ \dfrac{1}{j+p} \displaystyle\prod_{\alpha \in F} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1}\left[\left(T_{(\alpha,n)}\{(\psi_p \circ \|\cdot\|_j)'(x)\} \right)^2 \right] \right) \\ \qquad \delta = (F,j,p) \in S_\gamma \times \mathbb{N}^2 \qquad (2) \\ \dfrac{1}{j+p} T_{(\alpha,n)}[(\psi_p \circ \|\cdot\|_j)'(x)] \qquad \delta = (\alpha,n,j,p) \in \Gamma_\gamma \times \mathbb{N}^3 \qquad (3) \\ \dfrac{1}{j+p}(\psi_p \circ \|x\|_j) \qquad \delta = (j,p) \in \mathbb{N}^2. \qquad (4) \end{cases}$$

We shall show that $h_\delta(\cdot)$ is a C^k–smooth function from $X \to \mathbb{R}$, h^{-1} is continuous, and h is one–to–one, continuous and maps into $c_0(\Delta)$.To begin, we investigate the differentiability of the coordinates. For each $j \in \mathbb{N}$, $\|\cdot\|_j$ is $k+1$ times differentiable away from 0 and ψ_p is constant on a neighbourhood of 0. As noted above, their composites are $k+1$ times differentiable away from the origin and constant on a neighbourhood of the origin, so $(\psi_p \circ \|\cdot\|_j)$ is $k+1$ times differentiable at all $x \in X$

for all j and p. In addition, $T_{(\alpha,n)}$ is linear and the derivatives of $\tan^{-1} : \mathbb{R} \to \mathbb{R}$ vanish at infinity (so they are bounded) so it follows that the sum

$$\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1}\left[\left(T_{(\alpha,n)}\{(\psi_p \circ \|\cdot\|_j)'(x)\}\right)^2\right]$$

and each of its first k derivatives converges uniformly and thus it is k times differentiable. Since $I - \pi_\delta$ is C^∞–smooth we conclude that h_δ is at least a C^k–smooth function for each $\delta \in \Delta$.

Next, to show that h is 1–1, we suppose $h(x) = h(y)$ and consider two cases. If $x = 0$ then $(\psi_p\|\cdot\|_1)'(x) = 0$ because $\psi_p\|\cdot\|_1$ is constant on a neighbourhood of 0. Now T is 1–1 so by (3), $h(x) = h(y)$ forces $(\psi_p\|\cdot\|_1)'(y) = 0$. Consequently $\|y\|_1 \leq \frac{1}{p}$ for each $p \in \mathbb{N}$ and thus $y = 0$.

If x and y are not 0, then setting $c = \min(\|x\|, \|y\|))$ choose $P \in \mathbb{N}$ so that for $2P^{-1} < c$. By $(5-1)$, $\|x\|_j > c(1 - 1/2j)$ and $\|y\|_j > c(1 - 1/2j)$ so for $j \geq 4$ and $p > P$, $\psi_p \circ \|x\|_j = \|x\|_j$ and $\psi_p \circ \|y\|_j = \|y\|_j$. Using (4), $h(x) = h(y)$ implies $\|x\|_j = \|y\|_j$ for $j \geq 4$. In addition T is 1–1, so for $p > P$, $(\psi_p \circ \|\cdot\|_j)'(x) = \|\cdot\|_j'(x) = \|\cdot\|_j'(y)$. Therefore

$$\begin{aligned}\|x + y\|_j &\geq [\|\cdot\|_j'(x)](x + y) \\ &= [\|\cdot\|_j'(x)](x) + [\|\cdot\|_j'(y)](y) \\ &= \|x\|_j + \|y\|_j.\end{aligned}$$

Again by $(5-1)$ we have $\|x + y\| = \|x\| + \|y\|$. As noted in Section 2.2, WLUR implies strict convexity so $x = y$ and h is 1–1 as claimed.

We would now like to show that h maps X into $c_0(\Delta)$ so we fix $\epsilon > 0$ and $x \in X$. The key to this step of the proof is to ascertain the c_0–nature of the infinite sums which appear in lines (1) and (2). The first step in this direction is to show that for fixed j and p

$$1 \geq \sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1}\left[\left(T_{(\alpha,n)}\{(\psi_p \circ \|\cdot\|_j)'(x)\}\right)^2\right] \geq \epsilon \qquad (5-2)$$

for only finitely many $\alpha \in \Gamma_\gamma$. The upper bound is obvious, so to prove this inequality we let $K = \|(\psi_p\| \cdot \|_j)'(x)\|$, and choose $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2K^2}$. Now T maps into $c_0(\Gamma_\gamma \times \mathbb{N})$, so there exists a finite set $G \subset \Gamma_\gamma$ such that $T_{(\alpha,n)}(\psi\| \cdot \|_j)'(x) < \sqrt{\frac{\epsilon}{2}}$ for all $n \leq n_0$ and $\alpha \notin G$. Since $\|T\| \leq 1$, for all $\alpha \notin G$ we have

$$\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \Big(T_{(\alpha,n)} \{ (\psi_p \circ \| \cdot \|_j)'(x) \} \Big)^2 \leq \sum_{n=1}^{n_0} \frac{1}{2^n} \left(\frac{\epsilon}{2} \right) + \Big(\sum_{n=n_0+1}^{\infty} \frac{1}{2^n} \Big) K^2 < \epsilon.$$

Since $x \geq \tan^{-1}(x)$ for all positive reals, $(5-2)$ follows immediately.

We must now extend this observation to the case where there is a finite product of such terms. It should be noted that the norms occurring in these terms do not depend on each other and hence are arbitrary. With this in mind we fix an ordered, finite collection of natural numbers $\eta_j = (j_1, \ldots, j_n)$ and we fix $p \in \mathbb{N}$ and consider elements in S_γ^n of the form $\mathcal{F} = (F_1, \ldots, F_n)$. Using $(5-2)$ and appealing to Lemma 5.3, we know that for each $r, 1 \leq r \leq n$ there are only finitely many choices of $F \in S_\gamma$ satisfying

$$\prod_{\alpha \in F} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1} \left[\Big(T_{(\alpha,n)} \{ (\psi_p \circ \| \cdot \|_{j_r})'(x) \} \Big)^2 \right] \right) \geq \epsilon.$$

In addition, each factor of the term above is bounded above by 1 and thus

$$\prod_{\substack{\alpha \in F_r \\ 1 \leq r \leq n}} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1} \left[\Big(T_{(\alpha,n)} \{ (\psi_p \circ \| \cdot \|_{j_r})'(x) \} \Big)^2 \right] \right) \leq \epsilon$$

for all but a finite number of n–tuples $\mathcal{F} = (F_1, \ldots, F_n) \subset S_\gamma^n$.

We now use this information to show that there are only a finite number of indices as in (1) and (2) for which the coordinate value is greater than ϵ. For the case (2), if either of p or j are larger than ϵ^{-1} then $h_\delta(x) < \epsilon$; for those choices of j and p not in this category we apply the results above. For those indices as in (1)

we need a slightly more sophisticated approach. Firstly, $\psi_q \circ \|x\|_l \leq \max\{1, \|x\|_l\}$ and $\|x - \pi_s(x)\|_l \leq (1 + 1/l)\|x - \pi_s(x)\| \leq 4\|x\|$ so for each δ as in (1), $h_\delta(x) \leq \max\{\|x\|, 1\}$. Therefore, if any one of m, i, j, k, l, p, q are larger than $\epsilon^{-1} \max\{\|x\|, 1\}$ then $h_\delta(x) < \epsilon$. Now, if $\mathcal{F}$ is the second coordinate of $s = (\rho_m, \mathcal{F}, \eta_i, \eta_j, \eta_k) \in S$ then $\mathcal{F}$ must be of the form $\mathcal{F} = (F_1, \ldots, F_n)$ where $n = o(s)$ and for each choice of j, p both smaller than $\epsilon^{-1} \max\{\|x\|, 1\}$, and $\eta_j \in \mathbb{N}^n$ there are only finitely many choices of $\mathcal{F}$ satisfying

$$\prod_{\substack{\alpha \in F_r \\ 1 \leq r \leq n}} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1}\left[\left(T_{(\alpha,n)}\{(\psi_p \circ \|\cdot\|_{j_r})'(x)\} \right)^2 \right] \right) \geq \frac{\epsilon}{\max\{\|x\|, 1\}}.$$

In addition, the term on the left is bounded above by 1, thus if $\mathcal{F}$ contains any F not satisfying the above inequality then

$$\frac{\psi_q \circ \|x - \pi_s(x)\|_l}{4mijklqp} \prod_{\substack{\alpha \in F_r \\ 1 \leq r \leq o(s)}} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1}\left[\left(T_{(\alpha,n)}\{(\psi_p \circ \|\cdot\|_{j_r})'(x)\} \right)^2 \right] \right) \leq \epsilon.$$

Thus $h_\delta(x) \geq \epsilon$ for only a finite number of δ as in (1). For δ as in (3) and (4), the fact that $h_\delta(x) \geq \epsilon$ for only finitely many δ follows from similar arguments and thus h maps into $c_0(\Delta)$.

Because h is 1–1, $\{h_\delta(x)\}_{\delta \in \Delta}$ are equicontinuous and the range space of h is $c_0(\Delta)$ for some set Δ, Lemma 2.2 guarantees that h is continuous. Finally, we must show that h^{-1} is continuous. As in the previous chapter, we use Lemma 2.1 to prove this. Fix $\{x_m\}_{m=1}^{\infty}$ and x such that $h(x_m) \to h(x)$. If $x = 0$, then $\psi_p\|x\|_j = \frac{1}{2p}$ for each $j \in \mathbb{N}$, so for m large enough $\psi_p\|x_m\|_j < \frac{1}{p}$ and $x_m \to 0$. If $x \neq 0$ then $2 < p\|x\|$ implies $\psi_p(\|x\|_j) = \|x\|_j > \frac{1}{p}$ for each $j \in \mathbb{N}$, and $\psi_p(\|x_m\|_j) \to \|x\|_j$ forces $\|x_m\|_j \to \|x\|_j$. By (5−1) it then follows that $\|x_m\| \to \|x\|$ and thus $\{x_m\}_{m=1}^{\infty}$ is bounded.

To complete the proof we must show the second condition in Lemma 2.1 is satisfied. With this aim we fix $\epsilon > 0$ and use Lemma 5.1 to find $s = (\rho_m, \mathcal{F}, \eta_i, \eta_j, \eta_k) \in S$

with $F_r \subset \Lambda(\|\cdot\|'_{j_r}(x))$ such that $\|x - \pi_s(x)\| < \epsilon/4$. It follows from $(5-1)$ that $\|x - \pi_s(x)\|_1 < \epsilon/2$. Choosing $p > \frac{2}{\|x\|}$, we see that $(\psi_p\|\cdot\|_{j_r})'(x) = \|\cdot\|'_{j_r}(x)$. The choice of F_r ensures that

$$\prod_{\alpha\in F_r}\left(\sum_{n=1}^{\infty}\frac{1}{\pi 2^n}\tan^{-1}\left[\left(T_{(\alpha,n)}\{(\psi_p\circ\|\cdot\|_{j_r})'(x)\}\right)^2\right]\right) > 0;$$

$h_\delta(x_m) \to h_\delta(x)$ for each δ as in (2) implies there exists $M \in \mathbb{N}$ so that for each $m > M$

$$\prod_{\alpha\in F_r}\left(\sum_{n=1}^{\infty}\frac{1}{\pi 2^n}\tan^{-1}\left[\left(T_{(\alpha,n)}\{(\psi_p\circ\|\cdot\|_{j_r})'(x_m)\}\right)^2\right]\right) > 0.$$

Choosing $q > \frac{2}{\epsilon}$ we see that $\psi_q\|x - \pi_s(x)\|_1 = \|x - \pi_s(x)\|_1$. Assembling this information and noting in particular that $h_{(s,1,p,q)}(x_m) \to h_{(s,1,p,q)}(x)$ we see that, for the choice of p and q as above we get

$$\|x_m - \pi_s(x_m)\|_1 \to \|x - \pi_s(x)\|_1 < \frac{\epsilon}{2}.$$

Using (5–1) it is easy to show that for sufficiently large $m \in \mathbb{N}$ we have $\|x_m - \pi_s(x_m)\| < \epsilon$ and so for these m, x_m is within ϵ of the one dimensional subspace $\pi_s(X)$. This completes the proof.

5.3 Applications.

In this section, the focus will be to examine the various consequences to the theorem proved in the previous section. We begin by proving the following consequence of the theorem.

Corollary 5.1. *If X admits a C^{k+1}–smooth norm and if there exists a set Γ and a linear, bounded map $T : X \to c_0(\Gamma)$ with range of T^* dense in X^*, then X admits C^k–smooth partitions of unity.*

Proof: Let T be a linear, bounded map of X into $c_0(\Gamma)$. Using a result in [PWZ] we can find a sequence of C^∞–smooth norms $\|\cdot\|_n$ in $c_0(\Gamma)$ which converge

uniformly on bounded sets to a LUR norm $|\cdot|$. If $\|\cdot\|$ is a C^{k+1}–smooth norm on X, then we define a sequence of norms $\{|||\cdot|||_n\}_{n=1}^{\infty}$ on X by $|||x|||_n^2 = \|x\|^2 + \|Tx\|_n^2$. Since T is bounded $|||x|||_n$ is an equivalent C^{k+1}–smooth norm. In addition, note that

$$|||x|||_n^2 \to \|x\|^2 + |Tx|^2 = |||x|||^2.$$

Because $\|\cdot\|_n$ converges to $|\cdot|$ uniformly on bounded sets the same can be said of $|||\cdot|||_n$ with respect to $|||\cdot|||$. Clearly $|||\cdot|||$ is an equivalent norm which we claim to be WLUR.

To justify this claim suppose $2|||x|||^2 + 2|||x_m|||^2 - |||x + x_m|||^2 \to 0$. Because $2\|x\|^2 + 2\|x_m\|^2 - \|x + x_m\|^2 \geq (\|x\|^2 - \|x_m\|)^2$ and $2|Tx|^2 + 2|Tx_m|^2 - |T(x + x_m)|^2 \geq (|Tx|^2 - |Tx_m|)^2$, the latter expression must converge to 0. The LUR of the norm $|\cdot|$ implies that $Tx_m \to Tx$ in $c_0(\Gamma)$. Also $\|x_m\| \to \|x\|$ so that $\sup_m\{\|x - x_m\|\} = C < \infty$.

We shall show that if $\epsilon > 0$ and $f \in X^*$ are given, then there exists $M \in \mathbb{N}$ such that for all $m > M$, $|f(x_m - x)| < \epsilon$. Observe that because $T^* : c_0(\Gamma)^* \to X^*$ has dense range we can find an element $\theta \in c_0(\Gamma)^*$ such that $\|T^*(\theta) - f\| < \epsilon/2C$. Now there exists M such that $m > M$ implies $|\theta(T(x - x_m))| < \epsilon/2$. For such m

$$\begin{aligned} |f(x - x_m)| &\leq |(f - T^*\theta)(x - x_m)| + |T^*\theta(x - x_m)| \\ &\leq \|f - T^*\theta\|\|x - x_m\| + |\theta T(x - x_m)| \leq \epsilon \end{aligned}$$

Thus x_m converges weakly to x so $|||\cdot|||$ is WLUR. A direct application of Theorem 5.1 completes the proof.

Corollary 5.2. *Let X have a closed subspace Y such that Y^* is separable and X/Y has a shrinking M–basis. If X has a C^{k+1}–smooth norm, then X admits C^k–smooth partitions of unity.*

Proof: Clearly, from Corollary 5.1 it suffices to show that there exists a linear

bounded map $T : X \to c_0(\Gamma)$ (for some Γ) such that T^* has dense range. Let $(z_\alpha, z_\alpha^*)_{\alpha\in\Gamma_1}$ be a shrinking M–basis of $Z = X/Y$, let Q be the quotient map and by the natural isomorphism of Z^* with $Y^\perp$, let $h_\alpha = z_\alpha^* \circ Q$. Because Y^* is separable we can find a set $\{g_i\}_{i=1}^\infty$ dense on the unit sphere of Y^* and we extend these to $\{f_i\}_{i=1}^\infty$ in X^*. Now let Γ be the disjoint union of $\mathbb{N}$ and Γ_1, and we define $T : X \to c_0(\Gamma)$ by

$$T_\gamma(x) = \begin{cases} \frac{1}{i} f_i(x) & \gamma = i \in \mathbb{N} \\ h_\alpha(x) & \gamma = \alpha \in \Gamma_1. \end{cases}$$

Then T is linear and $\|T\|$ is bounded by one.

We now prove that T^* has dense range. Recall that e_γ is the element in $c_0(\Gamma)^*$ which has a one in the γ coordinate and zeros elsewhere. We show first that $T^* e_\alpha = h_\alpha$. Indeed if $x \in X$, then

$$(T^* e_\alpha - h_\alpha)(x) = e_\alpha[T(x)] - h_\alpha(x) = e_\alpha(h_\alpha(x)) - h_\alpha(x) = 0$$

and thus T^* maps onto a dense subset D of $Y^\perp$ by linearity. In addition, note that for each $x \in Y$, $(f_i - T^* i e_i)(x) = f_i(x) - i e_i[T(x)] = 0$ so $T^* i e_i = f_i$ on Y. Thus the element $i e_i$ is mapped by T^* to a member of the coset $f_i + Y^\perp$ say $f_i + h_i$. It is now easy to see that the linear span of elements of the form $f = f_i + h_i + d : i \in \mathbb{N}, d \in D$ is in the range of T^* and is dense in X^*.

Remarks 1. The space constructed in [CP] has a subspace isomorphic to $c_0(\Gamma_1)$ such that the resulting quotient is isomorphic to $c_0(\Gamma_2)$, yet the original space admits no linear bounded map into $c_0(\Gamma)$ for any Γ.

2. If X is WCG and has a C^k–smooth norm, then X has a shrinking M–basis [JZ]. It then follows from Corollary 5.2 and the proof of Corollary 5.1 that X admits a WLUR norm which is the uniform limit on bounded sets of C^k–smooth norms. Note that in general the space will not have a norm which is both C^k–smooth and LUR; the corresponding result for WLUR is open.

3. Recalling the properties of JL as listed in Chapter 4, we see that JL satisfies the conditions of Corollary 5.2 and thus we have another proof that JL admits C^∞–smooth partitions of unity.

Again we close with some questions. The theorem proved in this chapter assumes a smoothness criterion on a space and yields partitions of unity of one less degree of smoothness. Is it possible to obtain similiar results without losing a degree of smoothness? Also, in the previous chapter we only supposed the existence of a smooth function with nonempty bounded support. In light of the recent results in [H], the requirement that the approximations of the WLUR norm in Theorem 5.1 must be by smooth norms can be weakened to approximation by smooth functions with nonempty bounded support. This suggest two possible questions: 1) are smooth functions with nonempty bounded support sufficient to obtain the result?; 2) do the conditions of the theorem imply that C^k–smooth is equivalent to the existence of a C^k–smooth norm?

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